Applications of the functional renormalization group in cosmology and black hole thermodynamics

Thesis Submitted for the Degree of

Doctor of Philosophy (Science)

in

Physics (Theoretical)

by

Rituparna Mandal

Department of Physics

University of Calcutta

2022

Dedicated to my parents

Acknowledgements

First and foremost, I would want to convey my sincere gratitude to my supervisor Sunandan Gangopadhyay for his support, encouragement, and scientific guidance during my research. I have been inspired to study harder and put in my best effort by seeing his energy and enthusiasm in research. I also like to thank Amitabha Lahiri for his continuous encouragement and his meticulous suggestions throughout my research. I want to express my gratitude to all of the members in our research group, especially, Ankur, Debuda and my collaborators Soham, Sukantada. Moreover, I would like to thank my thesis committee for their challenging questions, and enlightening comments. Apart from that, I would like to say thanks to Anirban, Shantanu, Shauridi, Monalisadi, Subhasishda, Alikda, Dhirajda from SNBNCBS for many happy memories and for supporting me through numerous challenges.

In addition, I want to thank my friends Suchetana, Priti, Renu and Sneha from IISER Kolkata for many cheerful memories during my IISER days.

I want to acknowledge INSPIRE, Department of Science and Technology, Govt. of India for their financial assistance throughout my B.Sc. to Ph.D. academic path; without it, I would not have been able to continue my research.

Finally, I would like to express my indebtedness to my mother (Juthika Mandal), my father (Rabindranath Mandal) and my sister, Rima for their unconditional love, patience and constant encouragement throughout my journey.

সারসংক্ষেপ

আমি ফাংশনাল রিনর্মালাইজেশন গ্রুপ পদ্ধতিটি ব্যবহার করে, মহাবিশ্বতত্ত্ব (কসমোলজি) ও কৃষ্ণগহ্বরের (ব্ল্যাকহোল) তাপগতিবিদ্যা (থারমোডিনামিক্স) সংক্রান্ত কিছু মৌলিক প্রশ্নের উত্তর খোঁজার চেষ্টা করেছি এই প্রবন্ধে। স্টিভেন ওয়াইনবার্গ প্রথম "অ্যাসিম্পটোটিক সেফ গ্রাভিটি" তত্ত্বের উপস্থাপনা করেন, যা অতিবেগুনি শক্তি স্কেলে নন-গসীয়ান স্থির বিন্দুর (ফিক্সড পয়েন্ট) অস্তিত্বর কথা বলে। এই স্থির বিন্দুই অতিবেগুনি স্কেলে রিনর্মালাইজেশন প্রবাহকে অভৌতিক ডাইভারজেন্স থেকে রক্ষা করে। ফাংশনাল রিনর্মালাইজেশন গ্রুপ পদ্ধতির মূল উপাদান হল এফেক্টিভ অ্যাভারেজ অ্যাকশন যার প্রবাহ অতিবেগুনি শক্তি স্কেল থেকে নিম্ন শক্তি স্কেলে রিনর্মালাইজেশন গ্রুপ সমীকরণ দ্বারা নিয়ন্ত্রিত হয়। এর ফলস্বরুপ মহাকর্ষীয় ধ্রুবকের সাথে সাথে কসমোলজিকাল ধ্রুবকও সিস্টেমের শক্তি স্কেলের উপর নির্ভর করে পরিবর্তিত হয়। আমার প্রবন্ধের দ্বিতীয় অধ্যায়ে আমি ফাংশনাল রিনর্মালাইজেশন গ্রুপ পদ্ধতিটি কোয়ান্টাম মহাকর্ষের আইনস্টাইন-হিলবার্ট ট্রাঙ্কেশনের জন্য সংক্ষিপ্ত আলোচনা করেছি।

পরের দুটি অধ্যায়ে আমি, কোয়ান্টাম মহাকর্ষীয় প্রভাবকে অন্তর্ভুক্ত করার ফলস্বরূপ, বিয়াঙ্কি-I এবং FLRW মহাবিশ্বের কসমোলজিক্যাল ধারণার পরিবর্তন আলোচনা করেছি। বিয়াঙ্কি-I ইউনিভার্সের জন্য আমরা দিক নির্দেশক (ডিরেকশনাল) স্কেল ফ্যক্টর গণনা করেছি কোয়ান্টাম সংশোধিত মহাকর্ষীয় ধ্রুবক এবং কসমোলজিকাল ধ্রুবক দ্বারা তৈরি আইনস্টাইন সমীকরন থেকে। আমরা এই কাজে কন্সিসস্টেন্সি পদ্ধতিটি প্রয়োগ করেছি যেখানে এনার্জি মোমেন্টাম টেন্সর এবং আইনস্টাইন সমীকরণের ডানপক্ষের কোভ্যারিয়েন্ট সংরক্ষণ একসঙ্গে ব্যবহার করা হয়েছে। এখানে আমরা দেখেছি কোয়ান্টাম প্রভাবের কারনে বিয়াঞ্চি-I ইউনিভার্স FLRW ইউনিভার্সে পরিবর্তিত হয়েছে রেডিয়েশনের জন্য। কিন্তু ডাস্ট বা স্টিফ ম্যাটারের জন্য বিয়াঞ্চি-I ইউনিভার্স সাধারণত FLRW ইউনিভার্সে পরিবর্তিত হয় না। বরং স্টিফ ম্যাটারের জন্য বিয়াঞ্চি-I ইউনিভার্স কাজনার টাইপ সলিউসন দেখা যায়। এর পরবর্তী অধ্যায়ে আমরা দেখেছি কিভাবে FLRW ইউনিভার্সের ক্ষেত্রে কোয়ান্টাম সংশোধিত স্কেল ফ্যক্টরগুলি পরিবর্তিত হয় বিভিন্ন কাট-অফ স্কেলের উপর নির্ভর করে। এই কাজটিতে আমরা মডিফাইড কণ্টিনিউটি সমীকরণ পদ্ধতি ব্যবহার করেছি যা শুধুমাত্র কোয়ান্টাম সংশোধিত আইনস্টাইন সমীকরণের ডানপক্ষের কোভ্যারিয়েন্ট সংরক্ষণের উপর দাঁড়িয়ে আছে। এই কাজে আমরা দেখেছি যে বিভিন্ন কাট-অফ স্কেল বিভিন্ন লেট টাইম কসমোলজির আভাস দেয়।

পঞ্চম অধ্যায়ে আমরা কোয়ান্টাম সংশোধিত শোয়ার্জশিল্ড ব্ল্যাক হোলের তাপগতিবিদ্যার বৈশিষ্ট্য এবং কৃষ্ণগহ্বরের বাষ্পীভবন (ইভাপোরেশন) প্রক্রিয়া সম্পর্কে অনুসন্ধান করেছি। উল্লেখ্য এখানে কোয়ান্টাম

সংশোধিত শোয়ার্জশিল্ড সমাধান পেতে শোয়ার্জশিল্ড মেট্রিকে, নিউটনের মহাকর্ষ ধ্রুবকের পরিবর্তে, রিনর্মালাইজেশন গ্রুপ প্রবাহ থেকে প্রাপ্ত কোয়ান্টাম সংশোধিত মহাকর্ষীয় ধ্রুবক ব্যবহার করা হয়েছে। এই অধ্যায়ে আমরা দেখেছি যে কোয়ান্টাম সংশোধিত শোয়ার্জশিল্ড ব্ল্যাক হোলকে নির্দিষ্ট ব্যাসার্ধ বিশিষ্ট গোলাকার ক্যাভিটির মধ্যে রাখা যায় তাহলে এটিতে কখনও হকিং-পেজ দশান্তর (ফেজ টার্নজিশন) পরিলক্ষিত হয় না। অন্যদিকে ক্যাভিটির মধ্যে আবদ্ধ সাধারণ শোয়ার্জশিল্ড ব্ল্যাক হোলের ক্ষেত্রে একটি নির্দিষ্ট তাপমাত্রায় রেডিয়েশন স্টেট থেকে ব্ল্যাক হোলের স্টেটে হকিং-পেজ দশান্তর ঘটে থাকে।

অবশেষে ষষ্ঠ অধ্যায়ে আমরা, কোয়ান্টাম সংশোধিত শোয়ার্জশিল্ড ব্ল্যাক হোলে অবাধে পতনশীল পরমাণুর জন্য ত্বরণ বিকিরণ (অ্যাকসিল্যারেশন রেডিয়েশন) অনুসন্ধান করেছি। এখানে অবাধে পতনশীল পরমাণু ও বিকীর্ণ ফোটনের মধ্যে আপেক্ষিক ত্বরণের কারণে ত্বরণ বিকিরণের সৃষ্টি হয় যা অসীমে অবস্থিত পর্যবেক্ষকের কাছে হকিং বিকিরণের মতো দেখায়। যদিও ত্বরণ বিকিরণ হকিং বিকিরণের থেকে সম্পূর্ণ আলাদা। তাই ত্বরণ বিকিরণকে বেকেনস্টাইন-হকিং এনট্রপি থেকে আলাদা করার জন্য এই এনট্রপিকে হরাইজন ব্রাইটেনড ত্বরণ বিকিরণ (HBAR) এনট্রপি হিসাবে উল্লেখ করা হয়। আমরা কোয়ান্টাম সংশোধিত শোয়ার্জশিল্ড ব্ল্যাক হোলের জন্য HBAR এনট্রপি নির্ণয় করেছি এবং লক্ষ্য করেছি যে HBAR এনট্রপিতে বেকেনস্টাইন-হকিং এনট্রপি সদৃশ পদ ছাড়াও কোয়ান্টাম মহাকর্ষ সংশোধিত ব্ল্যাকহোল ক্ষেত্রফলের লগারিদমিক এবং বিপরীত বর্গমূল যুক্ত পদ রয়েছে।

Abstract

We investigate the cosmological implications and black hole thermodynamics employing the non-perturbative Functional Renormalization Group (FRG) approach in asymptotic safe gravity. Over the past few years, the functional renormalization group approach has offered a promising framework for creating a consistent and predictive quantum theory for gravitational interaction across all energy scales. The existence of an ultraviolet fixed point in the renormalisation group flow makes the theory asymptotically secure. In chapter 2, we briefly review the FRG techniques using the key object of modern FRG, the scale dependent effective average action. The functional renormalization group equation (FRGE) governs the evolution of gravitational action with the RG energy scale. This chapter provides a general framework for calculating the running cosmological constant and Newton's gravitational constant for the Einstein-Hilbert truncation solving the flow equations.

In the following two chapters, we have investigated the cosmological implications for Bianchi-I universe and FLRW universe taking into account quantum gravitational correction in the above mentioned formalism of the exact renormalization group flow of the effective average action for gravity. In chapter 3, we obtain the quantum corrected scale factors in different directions improving the Einstein equations for the Bianchi-I model with the scale dependent G and Λ in the consistency approach where the covariant conservation of the energy momentum tensor is used along with the covariant conservation of the right hand side of Einstein equation. It is shown that the scale factors eventually evolve into FLRW universe for known matter like radiation. However, for dust and stiff matter we find that the universe need not evolve to the FLRW cosmology in general,

but can also show Kasner type behaviour for stiff matter. In chapter 4, the effect of the scale dependency on the FLRW universe accounting quantum gravitational correction is then discussed. In particular, we look at how the scale factor changes over time when using different infrared cut-off scales in the context of a modified continuity equation approach where unlike the consistency equation approach, it only requires the covariant conservation of the right hand side of the Einstein equation. The main observation made here is that different cut-off identifications results in different late time cosmologies.

In chapter 5, we have investigated thermodynamical properties and studied their evaporation process in detail improving the Schwarzschild geometry with the quantum corrected running gravitational constant identifying $G(r) \equiv G(k = k(r))$. Here we investigate the phase transition for the quantum corrected Schwarzschild black hole maintained in a concentric spherical cavity and found it never undergoes Hawking-Page phase transition because the black hole state always prevails for all temperatures. It contrasts sharply with the normal Schwarzschild black hole scenario that is enclosed in a cavity, where the Hawking-Page phase transition takes place.

Finally in chapter 6, we have investigated the phenomenon of acceleration radiation exhibited by an atom falling into a quantum corrected Schwarzschild black hole. We have observed that the excitation-probability of the atom with simultaneous emission of a photon satisfies the equivalence principle when we compare it to the excitation probability of a mirror accelerating with respect to an atom. Then we have calculated the horizon brightened acceleration radiation (HBAR) entropy for this quantum corrected black hole geometry. We observed that the HBAR entropy has the form identical to that of Bekenstein-Hawking black hole entropy along with universal quantum gravity corrections.

List of Publications

- **Chapter 3**

1. **Rituparna Mandal**, Sunandan Gangopadhyay, Amitabha Lahiri, *"Cosmology of Bianchi type-I metric using renormalization group approach for quantum gravity"*, Classical and Quantum Gravity, **37**, 065012 (2020). Link

- **Conference proceedings**

2. Sunandan Gangopadhyay, **Rituparna Mandal**, Amitabha Lahiri, *"Bianchi-I cosmology in quantum gravity"*, Springer Proceedings in Physics, **277**, 925-929 (2022), Contributed to the proceedings of XXIV DAE-BRNS High Energy Physics Symposium 2020. Link.

- **Chapter 4**

3. **Rituparna Mandal**, Sunandan Gangopadhyay, Amitabha Lahiri, *"Cosmology with modified continuity equation in asymptotically safe gravity"*, The European Physical Journal Plus, **137**, 1110 (2022). Link

- **Chapter 5**

4. **Rituparna Mandal**, Sunandan Gangopadhyay, *"Black hole thermodynamics in asymptotically safe gravity"*, General Relativity and Gravitation, **54**, 159 (2022). Link

- **Chapter 6**

5. Soham Sen, **Rituparna Mandal**, Sunandan Gangopadhyay *"Equivalence principle and HBAR entropy of an atom falling into a quantum corrected black hole"*, Physical Review D, **105**, 085007 (2022) . Link

- **Other related publications (Not included in the thesis)**

6. Soham Sen, **Rituparna Mandal**, Sunandan Gangopadhyay, *"Near horizon aspects of acceleration radiation of an atom falling into a large class of static spherically symmetric black hole geometries"*, Physical Review D, **106**, 025004 (2022). Link

7. **Rituparna Mandal**, Sukanta Bhattacharyya, Sunandan Gangopadhyay *"Rainbow black hole thermodynamics and the generalized uncertainty principle"*, General Relativity and Gravitation, **50**, 143 (2018). Link

8. Sunandan Gangopadhyay, Biplab Paik, **Rituparna Mandal**,*"Accretion onto a noncommutative-inspired Schwarzschild black hole"*; International Journal of Modern Physics A, **33**, 1850084 (2018). Link

Table of contents

List of figures

Chapter 1

Introduction and Overview

General relativity is extremely successful as the theory of low energy, long distance gravitational interactions. The detection of gravitational waves from colliding black holes and neutron stars is in agreement with general relativistic predictions at an extraordinary level of instrumental precision, as are observations of the dynamics of binary pulsars, transfer of time measurements to atomic clocks in satellites of the Global Positioning System (GPS), and many other observations in the solar system as well as of astronomical objects (for a review of tests of general relativity and the agreement between theory and experiment, see [1, 2]). However, the presence of black hole singularities where matter is crushed by gravitational tidal forces as it falls into the black hole's centre signify the impossibility to determine how spacetime evolves beyond these singularities. The classical explanation of gravity in the setting of General Relativity fails at small scales because of these singularities [3]. A complete theory of quantum gravity is anticipated to be necessary in order to eliminate the unphysical divergences and restore predictability. The event horizon, however, prevents the breakdown of classical theory for an observer who is outside the black hole. As a result, the loss of information is censored by the event horizon, so the outside observer can only describe the black hole using a finite set of parameters. It is believed that a quantum theory of gravity would shed light on the regime beyond the event horizon.

Beyond pure classical gravity, there is also the semi-classical approach, which considers gravity as a classical theory while quantizing the other interactions of matter fields. Serious implications for black hole thermodynamics result from this approach. In particular, our comprehension of general relativity and its connection to quantum field theory has been significantly changed by the astonishing finding that black holes act like thermodynamic objects. In the early 1970s, Bekenstein [4, 5] and Hawking [6, 7] demonstrated that black holes radiate as black bodies with a characteristic temperature and entropy

$$T = \frac{\hbar\kappa}{2\pi k_B}, \quad S = \frac{A}{4\hbar G_0} \tag{1.1}$$

where κ and A are the surface gravity and area of the horizon respectively. These findings indicate that black hole thermodynamics should be investigated in the context of quantum gravity since both Planck's constant $\hbar$ and Newton's gravitational constant G_0 appear in the expression of entropy. Another clue comes from the information loss paradox, which says that a black hole initially created by the collapse of matter in a pure quantum state seems to not contain any information of the initial matter when observed by a distant observer as Hawking radiation. Since it would go against the unitarity of evolution, it is forbidden by standard quantum physics. What will happen to the information paradox if there are real quantum-gravitational effects? It is an obvious question to raise.

The three fundamental interactions of nature, namely electromagnetism and the strong and weak nuclear forces, are successfully described by quantum theory. These interactions are defined as gauge theories within the context of quantum field theory (QFT). There are many formal similarities between general relativity and non-Abelian gauge theories, starting from the fact that both are field theories based on local symmetries (see e.g. [8–12]) . Unlike non-Abelian gauge theories however, no consistent quantization of general relativity is known. The gravitational coupling is the dimensionful Newton's constant $G_0 \sim \frac{1}{M_{Pl}^2}$, while for gauge theories the coupling constants are dimensionless. So at each loop order in perturbative quantum gravity, the ultraviolet divergence is worse than in gauge theories by two powers of loop momentum. As a result, the number of distinct counterterms required to renormalize quantum gravity can be expected to be infinite, whereas the number of counterterms for a perturbatively renormalizable quantum gauge theory is finite.

The failure of perturbative methods to consistently quantize gravity does not however mean that a quantum theory of gravity cannot exist. It is possible for example that nonperturbative approaches such as loop quantum gravity [13, 14] may lead to a consistent quantum theory, or that by embedding gravity in a bigger theory such as supergravity [15, 16] or string theory [17–19] it may be possible to find a consistent quantum description of gravitation in four spacetime dimensions. Another possibility is that general relativity cannot be quantized but emerges as an effective field theory at low energies, thus including all possible diffeomorphism-invariant local functions of the metric which are not ruled out by other symmetries [20–22].

Yet another approach is to assume that the quantum theory which describes gravity in four dimensions is an "asymptotically safe" theory, i.e. the essential couplings of the theory hit a fixed point as the scale at which they are calculated is taken to infinity [23]. What this means is the following. Among all the couplings of the theory there are some "inessential couplings" Z for which $\dfrac{\partial \mathscr{L}}{\partial Z}$ is either zero or a total derivative when the field equations are satisfied. An example is the wave function renormalization constant, which can be eliminated by redefining the fields. The remaining coupling constants are the essential couplings, which will flow with an external parameter k which has the dimensions of mass. The meaning of this k depends on physics of the problem – it can be the momentum transfer in a scattering problem, or the inverse of some length scale specific to the problem. If the essential couplings g_i are dimensionful, we make them dimensionless by multiplying with suitable powers of k. Then the k-dependence of the essential couplings g_i are characterized by the $\beta_i(g) = k\dfrac{dg_i}{dk}$, called the β-functions. In general the β-function for any of the g_i will depend on all of the essential couplings. The β-functions of any theory describe a trajectory on the space of coupling constants. If a theory has a fixed point g^* in the space of all the g_i, the beta functions must vanish at that point, and also the trajectory for the theory must hit the point g^*. The trajectories which hit the fixed point form a hypersurface called the "critical surface", and asymptotically safe theories are defined to be those for which the couplings lie on the critical surface of a fixed point. Many non-Abelian gauge theories have ultraviolet fixed points where the gauge coupling vanishes, making the theory asymptotically free [24, 25]. For gravity, several calculations based on truncated exact Functional Renormalization Group (FRG) Equations support the conjecture that there is an interacting fixed point or Non-Gaussian fixed point (NGFP) in the ultraviolet regime [26–34]. Support for the conjecture also comes from calculations for gravity with matter or a cosmological constant [35–41]. The existence of an interacting UV fixed point implies there is an asymptotically safe quantum theory of gravity. Thus, the Asymptotic Safety conjecture can be summed up as follows: in the theory space of diffeomorphism invariant metric, gravity is a finite and predictive quantum field theory whose continuum limit is controlled by a finite-dimensional critical surface. So, if gravity were to be quantized, the result would be a (non-pertubatively) renormalizable quantum theory, whose high-energy completeness is specified by an NGFP. The corresponding quantum theory for asymptotic safety scenario is frequently referred as Quantum Einstein Gravity (QEG).

This doctoral dissertation focuses on two distinctively important aspects employing the (non-perturbative) Functional Renormalization Group (FRG) approach in asymptotic safe gravity. Firstly, we will explore cosmological implications of Asymptotic Safety and formulate predictions that may be tested against observations [42–44]. Secondly, we will assess the impact of the leading quantum gravity corrections on the properties of black hole thermodynamics within the framework of quantum Einstein gravity [45, 46].

In the FRG approach, the effective low energy theory is determined by solving an exact functional renormalization group (RG) flow equation and depends on a momentum shell parameter k. This results in the running of the coupling constants in the theory. The theory that comes from this formulation has its bare action corresponding to a non-trivial fixed point of the RG flow. When the approach is applied to gravity, a scale-dependent RG equation is derived by including all possible diffeomorphism invariant local functions of the metric [26, 33, 47–49]. This results in a running Newton's constant as well as running cosmological constant. Einstein equation is then "improved" by replacing the Newton's constant and cosmological constant by their scale-dependent versions. In the context of cosmology, this scale may be taken to be a function only of the cosmological time, so the running constants become time-dependent. However, the notion that the gravitational couplings could be affected by cosmic time and that this time-dependence might have consequences for cosmology is not new. In the context of RG-improved cosmologies this time-dependence emerges from the RG running of the gravitational couplings [50–62]. Here, we have investigated some of the main cosmological implications for Bianchi type-I metric and Friedmann-Lemaître-Robertson-Walker (FLRW) metric in functional renormalization group approach.

It is generally accepted that a semi-classical explanation of the formation and decay of black holes hold true as long as curvature effects remain small and the black hole mass is large compared to the Planck scale. The standard semi-classical scenario also states that a black hole with mass M emits Hawking radiation at a temperature that is inversely proportional to M. A number of unusual physical events, such as breaches of the conservation of baryon and lepton numbers or the "information paradox," could take place in the event of a full evaporation of a black hole [63]. In a coherent theory of quantum gravity, the issues of this kind should undoubtedly be addressed. For that, in the second part of my thesis, we have tried to understand the quantum gravitational effects in black hole thermodynamics in spherically symmetric black hole spacetimes using the renormalization group improvement method in asymptotic safe gravity framework, as it can clarify our understanding of the eventual fate of an evaporating black hole, in turns the famous "information paradox".

As we have already seen, black hole thermodynamics provides a coherent framework that unites the fundamental relationships between gravitation, quantum theory, and thermodynamics. After Bekenstein-Hawking entropy [4, 5] and Hawking radiation [6, 7], the thermal behaviour in accelerated systems discovered by Unruh, Davies, and Fulling [64–68] has been significant in helping us comprehend the thermodynamic framework that links black hole thermodynamics to acceleration radiation. One way to understand acceleration radiation is to consider how an accelerating atom in its ground state moving through the Minkowski vacuum would go to an excited state by absorbing Rindler particles [64], which would then be perceived by an inertial observer as the emission of Minkowski particles [65], or acceleration radiation. In that connection, it was demonstrated that another approach to understand Unruh acceleration radiation involves virtual processes in which atoms jump to an excited state while emitting a photon [69, 70]. However, we may make the virtual photons real by breaking and interrupting the virtual processes that are occurring all around us. The use of quantum optics to mimic the accelerated detector by a two-state atom is a relatively recent advancement in our understanding of the Unruh effect. Recent research has demonstrated that thermal radiation is experienced by an atom that is freely falling through the Boulware vacuum of a Schwarzschild black hole [71]. This appears to go against the equivalence principle at first look because the freely falling atom is in a locally inertial frame. However, it is the relative acceleration between the field modes and the freely falling atom produces the acceleration radiation. Moreover, under the right initial conditions, this acceleration radiation for atoms freely falling towards a black hole appears very similar to Hawking BH radiation to a distant observer. But the physics is completely different. In this case, the radiation is generated by the freely falling atoms cloud in Boulware Vacua, whereas Hawking radiation doesn't need any more matter. To distinguish it from the Bekenstein and Hawking BH entropy, this entropy is referred to as horizon brightened acceleration radiation (HBAR) entropy. Through this illustration of HBAR entropy flux, the existence of universal thermodynamic relations that are inherent to the black hole is made clear. The Einstein principle of the equivalent nature of gravity and acceleration in a small enough region of spacetime is also clarified by this analysis. It is necessary to investigate the equivalence principle for Hawking radiation or Unruh radiation as the quantum effects of this radiation has some inherent nonlocality [72]. Now, our objective is to investigate the status of the equivalence principle and HBAR entropy when an atom enters a renormalization group "improved" black hole spacetime by including pure quantum gravitational corrections in the setting of asymptotic safe gravity. More specifically, we seek to identify any

quantum gravity corrections in the HBAR entropy and determine if they share the same logarithmic properties as the Bekenstein-Hawking entropy's corrections. For this study, the entire sixth chapter of my thesis is devoted.
As mentioned above, the central theme of the thesis is to study some applications of functional renormalization group in cosmology and the black hole thermodynamics. We explore how quantum gravitational corrections have consequences in various physical systems. The outline of this thesis is as follows.

- **Chapter 2:** In this chapter, we present a review of the fundamental idea of the Effective Average Action and specifically discuss the Functional Renormalization Group (RG) Equation it generates. We first select the most basic configuration feasible, a scalar field theory on a non-dynamical, flat Euclidean space, to minimise unneeded technological complexities. We then move on to focus on the non-perturbative functional renormalization group technique for deriving the flow equation in the context of gravity. After that, we investigate the solutions of the flow equations for Newton's constant G and cosmological constant Λ at next to leading order in the infrared cutoff scale. A brief review is also provided for the solution of Newton's constant by letting the dimensionless cosmological constant $\lambda \approx 0$.

- **Chapter 3:** By accounting the quantum gravitational corrections in the formalism of the exact renormalization group flow of the effective average action for gravity, this chapter validates the investigation of the anisotropic Bianchi type-I cosmological model in late times. The cosmological evolution equations are derived by including the scale dependence of Newton's constant G and cosmological constant Λ. Using these scale dependent G and Λ in Einstein equations for the Bianchi-I model, we then investigate the scale factors in different directions to demonstrate how the quantum corrections affect Bianchi-I universe. This chapter is based on our work [42, 43].

- **Chapter 4:** In this chapter, we go through the detailed study of FLRW cosmology in functional group approach based on our work [44]. We investigate the quantum corrected scale factor, energy density, and entropy production at late times, taking different choices of cut-off functions. This approach is in line with previous studies which involve the running Newton constant $G(k)$ in the definition of energy-momentum tensor, and then imposing the covariant conservation identity of the Einstein tensor. The quantum corrections obtained in this approach are different from what are found by letting the conservation equation remain the same as for a scale-independent Newton constant.

- **Chapter 5:** In this chapter, we have investigated the black hole thermodynamics and the phase transition for renormalized group improved asymptotically safe Schwarzschild black hole geometry (based on [45]). This geometry takes into account the quantum gravitational correction in the running gravitational constant identifying $G(r) \equiv G(k = k(r))$. We study various thermodynamic quantity like the Hawking temperature, specific heat and entropy for the general parameter γ for quantum corrected Schwarzschild metric. We further investigate the local temperature, thermal stability of the black hole and the free energy considering a cavity enclosing the black hole. We also examine the final state of the quantum corrected black hole calculating the mass evaporation with time. It can give us a possible hint to understand the information loss paradox.

- **Chapter 6:** Here, we investigate the phenomenon of acceleration radiation exhibited by an atom falling into a quantum corrected Schwarzschild black hole. We have carried out calculation for the horizon brightened acceleration radiation (HBAR) entropy for this quantum corrected black hole geometry. We observe that the HBAR entropy has the form identical to that of Bekenstein-Hawking black hole entropy along with universal quantum gravity corrections. This chapter is based on our work published in Physical Review D [46].

- **Chapter 7:** Here, we conclude and summarize our thesis.

Chapter 2

The Functional Renormalization Group

The idea of the Renormalization Group (RG) has influenced the two fundamental branches of modern physics, namely, statistical mechanics and quantum field theory. Renormalization was first discussed in relation to quantum electrodynamics as a mathematical tool to resolve the unphysical ultraviolet divergences coming from loop to the transition amplitude. When a finite number of renormalized coupling constants are used to absorb these divergences replacing the bare couplings, a new effective theory is produced, whose coupling constants vary depending on the energy scale at which they are measured. In instance, the first Renormalization Group equation for quantum electrodynamics was developed in 1954 by Gell-Mann and Low explaining how the electric charge changes with the energy scale [73]. It actually closes the gap between the micro and macro scales by treating changes gradually from one scale to the next. In 1966, Kadanoff put out a plausible explanation for the critical phenomena to the discrete block spin RG in condensed matter physics which demonstrates how various physical systems behave similarly close to the critical point [74]. The development of the Renormalization Group theory was driven by this idea. The fundamental idea underlying the RG is the application of a coarse graining technique by averaging over fluctuations at short distance scales in order to immediately derive a description at longer distances from the microscopic principles.

After 1970s, the modern formulation of the Renormalization Group was created as a result of Wilson's [75, 76], as well as Kadanoff, and Fisher's [77, 78] development of a set of non-perturbative methods to examine the role of fluctuations in both quantum and thermal field theories. One of them, the non-perturbative functional RG method, enables research into the strongly-interacting regimes that characterise physics far from the Gaussian fixed point (GFP). In particular, Wilson and Kogut proposed in a paper [79] published in 1974 that the high-energy completion of a Quantum Field Theory can be defined if a fixed point attracts the renormalization group flow in the ultraviolet (UV) limit. For gravity, Winberg proposed that UV completion is possible if there is a non-trivial fixed point that protects the system from UV divergences [80]. This framework is known as asymptotically safe gravity theory.

The crucial tool of the functional RG is the flow equation which determines the evolution of the generating functional. It connects the macroscopic quantity integrating out the fluctuation with the microscopic physics. As a result, solving the flow equation is equivalent to solving the entire theory.

Field theories with gauge symmetry are especially important because they describe all elementary particle-physics interactions and gravity. A thorough understanding of gauge theories thus necessitates the use of a variety of field theoretical methods. First, Wettrich developed a functional method for scalar field theory that includes the Effective Average Action (EAA) as a key component. A Functional Renormalization Group Equation (FRGE) for scalar field theories has been derived, which is known as the Wetterich equation [81, 82]. The eventual generalisation of the EAA formalism to gauge theories [83] ignited interest in the possibility of discovering a NGFP. M. Reuter developed an exact equation describing the renormalization group flow of gravitational interaction in 1996 [26]. In particular, for gravity treating the metric fluctuations gradually from scale to scale rather than all at once, the search for non Gaussian fixed point (NGFP) has been reported in various different works including the Einstein-Hilbert approximation [29, 47, 84], with various matter fields [85–89], with running couplings in the ghost sector [90, 91], from Weyl invariant flows [92], in F(R) gravity [93–95], in approximations with higher derivatives [48, 96–98] and many.

In this chapter, we shall review the functional RG techniques with EAA formalism for scalar field theories. The EAA formalism will next be applied to the situation of quantum gravity, with a summary of the fundamental findings made by Reuter in [26, 99].

2.1 Functional RG approach to QFT

2.1.1 Standard effective action in QFT

In this subsection, we review the effective average action formalism in the functional renormalization group approach to quantum field theory following articles [100, 101]. There have also been several literary works that have made reference to the comprehensive research of effective average action formalism in functional RG [102–104]. We will first start by considering the real scalar field theory in flat Euclidean space-time without any gauge symmetries. In quantum field theory (QFT), the key object is the n-point correlation functions which are defined by the weighted averages of the product of n fields at different space-time points over all possible field configurations

$$\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)...\varphi(x_n)\rangle = \frac{\int \mathcal{D}\varphi\ \varphi(x_1)\varphi(x_2)\varphi(x_3)...\varphi(x_n)e^{-S[\varphi]}}{\int \mathcal{D}\varphi\ e^{-S[\varphi]}} \quad . \tag{2.1}$$

The above correlation function is weighted with an exponential of the action $S[\varphi]$ for every field configuration. These n-point functions can be easily derivable by defining the generating functional $Z[J]$ which is functional of the external source J

$$Z[J] \equiv e^{W[J]} = \int \mathcal{D}\varphi e^{-S[\varphi]+J.\varphi} \tag{2.2}$$

where $J.\varphi \equiv \int d^dx J(x).\varphi(x)$ and the dot denotes the Euclidean space integral. Now all n-point functions are obtained as a functional derivative of $Z[J]$ with respect to the source term J

$$\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)...\varphi(x_n)\rangle = \left(\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right)_{J=0} . \tag{2.3}$$

Another generating functional $W[J]$ of all connected correlation functions satisfying $W[J] = \ln Z[J]$ in Euclidean space, have been also introduced in eq.(2.2). A third kind of generating functional, $\Gamma[\phi]$ which is known as the the standard effective action, is obtained by Legendre transformation of $W[J]$

$$\Gamma[\phi] = \sup_J \left(\int d^d x J(x)\phi(x) - W[J]\right) . \tag{2.4}$$

The effective action, $\Gamma[\phi]$ is the functional of the field expectation value $\phi(x) \equiv \langle\varphi(x)\rangle_J = \frac{\delta W[J]}{\delta J(x)}$ in presence of the source term J. Here, $\sup_J$ means that the supremum value of J is used to take the RHS of (2.4) at $J = J_{sup}[\phi]$, making the resultant action a functional of the field. The functional Γ generates all the one-particle irreducible diagram by functional differentiation with respect to $\phi(x)$. The quantum equation of motion for ϕ can be derived taking the functional derivative of $\Gamma[\phi]$ with respect to $\phi(x)$

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = J(x) . \tag{2.5}$$

This equation of motion captures all the effects of quantum fluctuations and governs the dynamics of the field expectation value $\phi(x)$. Substituting $W[J]$ from eq.(2.4) in eq. (2.2), that is, in equation $Z[J] = e^{W[J]}$, the effective action can be given in a functional integral representation as

$$e^{-\Gamma[\phi]} = \int \mathcal{D}\chi \exp\left\{-S[\phi+\chi] + \frac{\delta\Gamma[\phi]}{\delta\phi}.\chi\right\} \tag{2.6}$$

where the shift in integration variable is defined as $\varphi = \phi + \chi$. The fluctuations χ around the background field ϕ is treated as perturbation for obtaining Dyson-Schwinger equation by a vertex expansion of $\Gamma[\phi]$.

2.1.2 Effective average action (EAA)

The effective average action, $\Gamma_k[\phi]$, is the generalization of the effective action $\Gamma[\phi]$ which has built-in Infrared (IR) cutoff at k. The effective average action, Γ_k plays the role of coarse grained free energy in the coarse gaining length k^{-1} in statistical physics. This IR scale uses to identify fluctuations with momenta $p^2 \geqslant k^2$ which are included in the Γ_k. The more and more fluctuations should be included in the effective action successively if we lower k step by step. This results to explore the theory in larger length scales. By construction, the effective action interpolates between the ordinary effective action, Γ_0 at $k = 0$ where all the fluctuations are included and the microscopic or classical action S for very large energy scale $k \to \Lambda$

$$\Gamma_{k\to\Lambda} \simeq S, \quad \Gamma_{k\to 0} = \Gamma_0[\phi] \ . \tag{2.7}$$

Here no fluctuations are effectively included when the IR cutoff k goes to the physical ultraviolet cutoff Λ.

Now the generating functional for the connected correlation function in the presence of an IR cutoff can be written in the path integral formalism

$$W_k[J] = \ln Z_k[J] = \ln \int \mathcal{D}\varphi \ e^{-S[\varphi]-\Delta S_k[\varphi]+J.\varphi} \tag{2.8}$$

where

$$\Delta S_k[\varphi] = \frac{1}{2}\varphi.\mathcal{R}_k(-\partial^2).\varphi \tag{2.9}$$

is a regulator term which is quadratic in φ. The IR cutoff function, $\mathcal{R}_k$ can be regarded as a momentum dependent mass term. This $\mathcal{R}_k$ must satisfy two conditions in momentum space as

$$\mathcal{R}_k(p^2) = k^2 \quad for \quad p^2 \ll k^2, \quad \mathcal{R}_k(p^2) = 0 \quad for \quad p^2 \gg k^2 \ . \tag{2.10}$$

The IR cutoff function, $\mathcal{R}_k$ suppresses all the fluctuations with momentum $p^2 \ll k^2$ in a mass like fashion. On the other hand, the IR cutoff function, $\mathcal{R}_k$ vanishes for $p^2 \gg k^2$, so that all the fluctuations above k are included in the effective average action Γ_k.
The scale dependent expectation value of φ, that is, ϕ can be written from eq.(2.8) due to the presece of IR cutoff function

$$\phi \equiv \langle \varphi \rangle = \frac{\delta W_k\,[J]}{\delta J(x)}\ . \tag{2.11}$$

It should be noted that the microscopic field ϕ and the source term J are scale dependent, so that $\phi = \phi_k[J]$ and therefore $J = J_k[\phi]$.
Next, the Legendre transformation of W_k is defined as

$$\tilde{\Gamma}_k\,[\phi] = \sup_J \left(\int d^d x J(x)\phi(x) - W_k[J] \right)\ . \tag{2.12}$$

The scale dependent effective average action (EAA), $\Gamma_k[\phi]$ is obtained subtracting the IR regulator term $\Delta S_k[\phi]$,

$$\Gamma_k[\phi] = \tilde{\Gamma}_k[\phi] - \frac{1}{2}\phi.\mathcal{R}_k(-\partial^2).\phi\ . \tag{2.13}$$

The reason behind the substruction of $\Delta S_k[\phi]$ is that the functional $\tilde{\Gamma}_k[\phi]$ does not possess the correct asymptotic behaviour in the limit $k \to \infty$. Now, it can be seen that the effective average action, Γ_k reach at the standard effective action $\Gamma[\phi]$ given in (2.7) in the limit $k \to 0$ since the cutoff function $\mathcal{R}_k$ vanishes for all the p^2 modes in this limit. To show the UV limit, we differentiate the EAA with respect to the scale dependent average field and recover the identity

$$\frac{\delta \Gamma_k[\phi]}{\delta \phi(x)} = J(x) - (\mathcal{R}_k.\phi)(x)\ . \tag{2.14}$$

Shifting the integration variable to the fluctuation field χ around the the coarse gained field as $\chi = \varphi(x) - \phi(x)$, the expression for the EAA, Γ_k can be obtained using the same procedure for the standard effective action as described before in (2.6)

$$e^{-\Gamma_k[\phi]} = \int \mathcal{D}\chi \exp\left\{-S\left[\phi+\chi\right] + \frac{\delta\Gamma_k[\phi]}{\delta\phi}.\chi - \frac{1}{2}\chi.\mathcal{R}_k.\chi\right\} . \tag{2.15}$$

The term $\exp\left\{-\frac{1}{2}\chi.\mathcal{R}_k.\chi\right\}$ behaves as a delta functional $\delta[\chi]$, since the regulator term $\mathcal{R}_k$ diverges in the UV limit $k^2 \to \infty$. Thus, the effective average function, Γ_k satisfies $\Gamma_k \approx S$ described in (2.7) in the UV limit, as it yields $\exp\{-S[\phi]\}$ on the right hand side of eq.(2.15) performing the integral over χ. In short, the effective average action $\Gamma_k[\phi]$ enables us to connect between the macroscopic physics explained by the effective action $\Gamma[\phi]$ including all quantum fluctuations in the effective action and the microscopic physics represented by the bare action S.

2.1.3 The Wetterich equation

To calculate the effective action, one can follow two approaches. One approach leads to Dyson-Schwinger equation computing $\Gamma[\phi]$ term by term in a vertex expansion. Here all the fluctuations are integrated out at once. Another approach involves the Wilson's idea of integrating out quantum fluctuations of momentum shell at a time. Here the scale dependent effective average action Γ_k depends on the coarse gained scale k and follows the exact non-perturbative functional renormalization group equation (FRGE) or flow equation. The dependence of Γ_k on the scale k is described by

$$k\frac{\partial}{\partial k}\Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} k\frac{\partial}{\partial k}\mathcal{R}_k\right] \tag{2.16}$$

where the inverse average propagator or Hessian $\Gamma_k^{(2)}$ is given by the second functional derivative with respect to the expectation value of field

$$\Gamma_k^{(2)}(x,y) \equiv \frac{\delta^2\Gamma_k}{\delta\phi(x)\delta\phi(y)} . \tag{2.17}$$

Here the trace involves an integration over momentum or coordinates and the cutoff operator has matrix element $\mathcal{R}_k\,(x,y) \equiv \mathcal{R}_k\,(-\partial_x^2)\,\delta\,(x-y)$.

To derive the exact flow equation (2.16), one can start by taking the scale derivative of $\tilde{\Gamma}_k[\phi]$ at constant field, stated by (2.12)

$$\frac{\partial}{\partial k}\tilde{\Gamma}_k[\phi] = -\left(\frac{\partial W_k}{\partial k}\right)[J]\Bigg|_J - \int \mathrm{d}^d x\,\frac{\delta W_k}{\delta J(x)}\frac{\partial J(x)}{\partial k} + \int \mathrm{d}^d x\;\phi(x)\frac{\partial J(x)}{\partial k}\;. \tag{2.18}$$

The last two terms of the above equation cancel each other as $\phi(x) = \frac{\delta W_k}{\delta J(x)}$. For fixed J, that is for the scale independent J, the k-derivative is obtained from the functional integral (2.8)

$$\begin{aligned}\left(\frac{\partial W_k}{\partial k}\right)[J]\Bigg|_J &= -\frac{Z_k^{-1}}{2}\int \mathrm{d}^d x\int \mathrm{d}^d y\int \mathcal{D}\varphi\;\varphi(x)\frac{\partial}{\partial k}\mathcal{R}_k(x,y)\varphi(y)e^{-S[\varphi]-\Delta S_k[\varphi]+J.\varphi}\\ &= -\frac{1}{2}\int \mathrm{d}^d x\int \mathrm{d}^d y\,\langle\varphi(x)\varphi(y)\rangle\,\frac{\partial}{\partial k}\mathcal{R}_k(x,y)\;.\end{aligned} \tag{2.19}$$

Here the contribution comes only from $\Delta S_k[\varphi]$ term as $\mathcal{R}_k$ depends on the scale k. Substituting (2.19) in eq.(2.18), the scale dependence of $\tilde{\Gamma}_k$ can be written as

$$\begin{aligned}\frac{\partial}{\partial k}\tilde{\Gamma}_k[\phi] &= \frac{1}{2}\int \mathrm{d}^d x\int \mathrm{d}^d y\left\{\frac{\partial}{\partial k}\mathcal{R}_k(x,y)G_k(y,x) + \phi(x)\frac{\partial}{\partial k}\mathcal{R}_k(x,y)\phi(y)\right\}\\ &\equiv \frac{1}{2}\mathrm{Tr}\left\{G_k\frac{\partial}{\partial k}\mathcal{R}_k\right\} + \frac{\partial}{\partial k}\Delta S_k[\phi]\;.\end{aligned} \tag{2.20}$$

In the above equation, we have used the connected two-point function defined as

$$G_k(x,y) \equiv \frac{\delta^2 W_k}{\delta J(x)\delta J(y)} = \langle\varphi(x)\varphi(y)\rangle - \langle\varphi(x)\rangle\,\langle\varphi(y)\rangle\;\;. \tag{2.21}$$

As the relation between ϕ and J was invertible, one can anticipate the source as a functional of the field $J = J[\phi]$ such that

$$\int d^d y G_k(x,y)\frac{\delta J(y)}{\delta\phi(z)} = \delta(x-z)\;. \tag{2.22}$$

Now using eq.(2.14), one can easily deduce that

$$\frac{\delta J(x)}{\delta\phi(y)} = \frac{\delta^2\Gamma_k[\phi]}{\delta\phi(y)\delta\phi(x)} + \mathcal{R}_k(x,y) \ . \tag{2.23}$$

Substituting the above equation in eq.(2.22), we can easily obtained the connected propagator by the inverse continuous matrix as

$$G_k(x,y) = \left[\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + \mathcal{R}_k\right]^{-1}(x,y) \ . \tag{2.24}$$

Now, to obtain the flow equation (2.16) at constant field, one can just evaluate the scale derivative of the flowing action k provided by (2.13) with (2.12)

$$\begin{aligned} k\frac{\partial}{\partial k}\Gamma_k[\phi] &= k\frac{\partial}{\partial k}\tilde{\Gamma}_k[\phi] - k\frac{\partial}{\partial k}\Delta S_k \\ &= \frac{1}{2}\mathrm{Tr}\left\{G_k \ k\frac{\partial}{\partial k}\mathcal{R}_k\right\} \\ &= \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} k\frac{\partial}{\partial k}\mathcal{R}_k\right] \ . \end{aligned} \tag{2.25}$$

The second line is obtained substituting eq.(2.20) in place of $\frac{\partial}{\partial k}\tilde{\Gamma}_k[\phi]$ in the first line of the above equation. This equation is also known as the Wetterich equation. Despite the fact that we have only obtained the equation for a conventional scalar field, it can be applied to fermions, gauge theories, and gravity.

2.2 RG equation for gravity

2.2.1 EAA for gravity

In this section we briefly review the effective average action formalism for Euclidean quantum gravity in d dimensions developed in [26]. The analysis is based on an (Euclidean) "effective average action" $\Gamma_k[g_{\mu\nu}]$ defined such that it correctly describes all gravitational phenomena, including the effect of all loops, at a momentum scale k. Even though the quantum effective action contains all the information about the quantum theory, it turns out that it is more convenient to work with an alternative functional called the effective average action [26, 29, 38, 95, 99, 105–108], which is calculated like the effective action but with an infrared cutoff at the scale k. Modes with $p^2 \ll k^2$ are excluded while those with $p^2 \gg k^2$ are integrated out in the usual way. The classical action S corresponds to ignoring all quantum modes, while the usual effective action Γ corresponds to removing the IR cutoff, so Γ_k interpolates between $S = \Gamma_{k\to\infty}$ and $\Gamma = \Gamma_{k=0}$. Then as a function of k this Γ_k describes a trajectory which satisfies a renormalization group flow equation. The flow equation then reads

$$\begin{aligned} k\partial_k \Gamma_k[g,\bar{g}] = \frac{1}{2}\mathrm{Tr}\left[\left(\kappa^{-2}\Gamma_k^{(2)}[g,\bar{g}] + \mathcal{R}_k^{grav}[\bar{g}]\right)^{-1} k\partial_k \mathcal{R}_k^{grav}[\bar{g}]\right] \\ - \mathrm{Tr}\left[\left(-M[g,\bar{g}] + \mathcal{R}_k^{gh}[\bar{g}]\right)^{-1} k\partial_k \mathcal{R}_k^{gh}[\bar{g}]\right]. \end{aligned} \tag{2.26}$$

We have also defined $\kappa = (32\pi\bar{G})^{-\frac{1}{2}}$, where $\bar{G}$ is the value of $G(k)$ as $k \to \infty$. The first trace on the right-hand side of the equation is defined due to the expectation value of metric fluctuations, $h_{\mu\nu} = \left\langle \hat{h}_{\mu\nu} \right\rangle$, and the second one comes from the Faddeev-Popov ghost. Here the equation is written down in terms of

$$\begin{aligned} \Gamma_k[g,\bar{g}] &= \Gamma_k^{grav}[g,\bar{g}] + S_{gf}[g-\bar{g};\bar{g}] \\ &= \bar{\Gamma}_k[g] + S_{gf}[g-\bar{g};\bar{g}] + \hat{\Gamma}_k[g,\bar{g}] \end{aligned} \tag{2.27}$$

where $\bar{\Gamma}_k[g] \equiv \Gamma_k^{grav}[g,g]$, and $\hat{\Gamma}_k[g,\bar{g}]$ contains all the deviations for $g \neq \bar{g}$. $\Gamma_k^{(2)}[g,\bar{g}]$ is the Hessian of $\Gamma_k[g,\bar{g}]$ with respect to the expectation value of the the full metric, $g_{\mu\nu} = \langle \hat{g}_{\mu\nu} \rangle$ (where $\hat{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \hat{h}_{\mu\nu}$) at fixed background $\bar{g}_{\mu\nu}$. Furthermore, M is the Faddeev-Popov ghost operator, while $\mathcal{R}_k^{grav}[\bar{g}]$ and $\mathcal{R}_k^{gh}[\bar{g}]$ are the IR cutoff functions for gravity and the ghost operator, respectively.

We will take both of these to be of the form $\mathcal{R}_k(p^2) \propto k^2 R^{(0)}(p^2/k^2)$ where the function $R^{(0)}(z)$ is smooth and satisfies the conditions $R^{(0)}(0) = 1$ and $R^{(0)}(z) \to 0$ for $z \to \infty$, but is otherwise arbitrary. In the calculation for Γ_k, the p^2 is replaced by the kinetic operator for gravitons or ghosts. Following [26] we will take $R^{(0)}(z)$ to be of the form

$$R^{(0)}(z) = z\left[\exp(z) - 1\right]^{-1} . \tag{2.28}$$

It is noted that other choices for the regulator function are possible [47, 109, 110]. However, as it is apparent below, the choice of different regulator functions do not qualitatively change the results.

The Einstein-Hilbert truncation: The use of the functional renormalization group equation (2.26) is illustrated using the single metric truncation where one can set $\hat{\Gamma}_k[g,\bar{g}]$ exactly to zero, or $\hat{\Gamma}_k[g,\bar{g}]$ can take same form as gauge fixing action S_{gf} with a different prefactor, that is

$$\hat{\Gamma}_k[g,\bar{g}] \equiv 0 \qquad \text{or} \qquad \hat{\Gamma}_k[g,\bar{g}] \propto S_{gf} . \tag{2.29}$$

Now, the effective average action is inspired from the classical Einstein-Hilbert truncation in d dimensions in the form

$$S = \frac{1}{16\pi\bar{G}} \int \mathrm{d}^d x \sqrt{g} \left(-R(g) + 2\bar{\Lambda}\right) \tag{2.30}$$

where $\bar{G}$ and $\bar{\Lambda}$ are the k-independent classical Newton's constant and cosmological constant at $k \to \infty$. Here, the scale dependence in these coupling constants can be determined solving the RG equation with an ansatz defined in (2.27) where $\hat{\Gamma}_k[g,\bar{g}] \propto S_{gf}$. Here, the infinite dimensional space of all action functionals are then projected on the 2-dimensional subspace spanned by the functions $\sqrt{g}$ and $\sqrt{g}R$ to obtain solutions to the RG equation. For this choice of truncation in the background metric formalism, we need to consider effective actions only of the form

$$\begin{aligned}\Gamma_k[g,\bar{g}] &= (16\pi G(k))^{-1}\int d^dx\sqrt{g}\,\{-R(g)+2\Lambda(k)\}+S_{gf}[g-\bar{g},\bar{g}]\\ &= 2\kappa^2 Z_{Nk}\int d^dx\sqrt{g}\,\{-R(g)+2\Lambda(k)\}\\ &\qquad + \kappa^2 Z_{Nk}\int \mathrm{d}^dx\sqrt{\bar{g}}\;\bar{g}_{\mu\nu}\,\left(\mathcal{F}_\mu{}^{\alpha\beta}g_{\alpha\beta}\right)\,\left(\mathcal{F}_\nu{}^{\rho\sigma}g_{\rho\sigma}\right)\end{aligned} \tag{2.31}$$

where $\kappa \equiv (32\pi\bar{G})^{-\frac{1}{2}}$ and $S_{gf}[g,\bar{g}]$ is a classical background gauge fixing term. Here, the choice of gauge is written in the harmonic gauge as

$$F_\mu[h;\bar{g}] = \sqrt{2}\kappa\;\mathcal{F}_\mu{}^{\alpha\beta}g_{\alpha\beta} = \sqrt{2}\kappa\;\mathcal{F}_\mu{}^{\alpha\beta}h_{\alpha\beta} = \sqrt{2}\kappa\left(\bar{D}^\nu h_{\mu\nu}-\frac{1}{2}\bar{D}_\mu\bar{g}^{\alpha\beta}h_{\alpha\beta}\right) \tag{2.32}$$

where $\mathcal{F}_\mu{}^{\alpha\beta}g_{\alpha\beta} = \mathcal{F}_\mu{}^{\alpha\beta}h_{\alpha\beta}$ and $\bar{D}_\mu g_{\alpha\beta}=0$. It is to be noted that the chosen $\hat{\Gamma}_k$ replaces the prefactor of the original S_{gf} term in (2.31) by the scale dependent ratio $\frac{\bar{G}}{G(k)}$ which can be referred as the dimensionless function $Z_{Nk}=\frac{\bar{G}}{G(k)}$. It is also worth mentioning that it is possible to truncate so as to include higher derivative invariants. However, for a three-dimensional subspace it is known that the flow is essentially two dimensional close to the fixed point. Further, the projected 2-dimensional flow gets nicely approximated by the Einstein-Hilbert flow [28].

2.2.2 Flow of G and Λ

To find the flow equation for scale dependent coupling constant, Z_{Nk} and $\Lambda(k)$ on the Einstein-Hilbert space were spanned by $\int\sqrt{g}$ and $\int\sqrt{g}R$, so the left hand side of the flow equation can be written having inserted the effective action in (2.26)

$$\partial_t\Gamma_k[g,g] = 2\kappa^2\int d^dx\sqrt{g}\,\{-R(g)\;\partial_t Z_{Nk}+2\partial_t\left(Z_{Nk}\Lambda(k)\right)\}\;. \tag{2.33}$$

Here $\bar{g}_{\mu\nu}=g_{\mu\nu}$ is set so that the gauge fixing term in (2.31) vanishes. Now for the right hand side, the derivative expansion have to be performed retaining only the term proportional to $\int\sqrt{g}$ and $\int\sqrt{g}R$. To obtain the second functional derivative of the effective action, $\Gamma_k^{(2)}$, one expands

$$\Gamma_k[\bar{g}+h,\bar{g}] = \Gamma_k[\bar{g},\bar{g}] + \mathcal{O}(h) + \Gamma_k^{quad}[h;\bar{g}] + \mathcal{O}(h^3) \tag{2.34}$$

where

$$\Gamma_k^{quad} = Z_{Nk}\kappa^2 \int \mathrm{d}^d x \sqrt{\bar{g}}\; h_{\mu\nu} \left[-K^{\mu\nu}{}_{\rho\sigma}\bar{D}^2 + U^{\mu\nu}{}_{\rho\sigma} \right] h^{\rho\sigma} \; . \tag{2.35}$$

The tensors K and U are given by

$$K^{\mu\nu}{}_{\rho\sigma} = \frac{1}{4}\left[\delta^\mu_\rho \delta^\nu_\sigma + \delta^\mu_\sigma \delta^\nu_\rho - \bar{g}^{\mu\nu}\bar{g}_{\rho\sigma} \right] \tag{2.36}$$

and

$$\begin{aligned} U^{\mu\nu}{}_{\rho\sigma} = & \frac{1}{4}\left[\delta^\mu_\rho \delta^\nu_\sigma + \delta^\mu_\sigma \delta^\nu_\rho - \bar{g}^{\mu\nu}\bar{g}_{\rho\sigma} \right] \left(\bar{R} - 2\Lambda(k) \right) + \frac{1}{2}\left[\bar{g}^{\mu\nu}\bar{R}_{\rho\sigma} + \bar{g}_{\rho\sigma}\bar{R}^{\mu\nu} \right] \\ & - \frac{1}{4}\left[\delta^\mu_\rho \bar{R}^\nu{}_\sigma + \delta^\mu_\sigma \bar{R}^\nu{}_\rho + \delta^\nu_\rho \bar{R}^\mu{}_\sigma + \delta^\nu_\sigma \bar{R}^\mu{}_\rho \right] - \frac{1}{2}\left[\bar{R}^\nu{}_\rho{}^\mu{}_\sigma + \bar{R}^\nu{}_\sigma{}^\mu{}_\rho \right] \end{aligned} \tag{2.37}$$

where $\bar{g}_{\mu\nu}$ is used for lowering and raising the indices.

Now to proceed further, the quadratic form of (2.35) is partially diagonalized by decomposing the metric $h_{\mu\nu}$ as

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + d^{-1}\bar{g}_{\mu\nu}\phi \; , \qquad \bar{g}^{\mu\nu}\bar{h}_{\mu\nu} = 0 \; . \tag{2.38}$$

In order to calculate the differential equation for the scale dependent quantity Z_{Nk} and $\Lambda(k)$ comparing $\int \sqrt{g}$ and $\int \sqrt{g}R$ from both side of eq.(2.26), the quadratic action is written by utilizing the freedom of maximally symmetric spacetime [9]

$$\begin{aligned} \Gamma_k^{(2)} = \frac{1}{2} Z_{Nk}\kappa^2 \int \mathrm{d}^d x \sqrt{\bar{g}} \Big\{ & \bar{h}_{\mu\nu}\left[-\bar{D}^2 - 2\Lambda(k) + C_T R \right] \bar{h}^{\mu\nu} \\ & - \left(\frac{d-2}{2d} \right) \phi \left[-\bar{D}^2 - 2\Lambda(k) + C_S R \right] \phi \Big\} \end{aligned} \tag{2.39}$$

where $C_T \equiv \frac{d(d-3)+4}{d(d-1)}$, $C_S \equiv \frac{d-4}{d}$.

Cutoff scheme: Prior to moving forward, it is necessary to select the cutoff operators that better describe the change from a high momentum to a low momentum regime. For this, it is necessary to modify the cutoff specified in eq.(2.9) to act on scalar field action, so that it fits in the current circumstances. There has already been some discussion in subsection (2.2.1). The convenient cutoff operator has structure

$$\mathcal{R}_k(\bar{g}) = \mathcal{Z}_k k^2 R^{(0)} \left(-\bar{D}^2/k^2\right) \tag{2.40}$$

where the matrix $\mathcal{Z}_k$ for graviton and ghost are given in the following form

$$\begin{aligned}(\mathcal{Z}_k^{grav})^{\mu\nu\rho\sigma} &= \left[(\mathbf{1} - P_\phi)^{\mu\nu\rho\sigma} - \frac{d-2}{2} P_\phi^{\mu\nu\rho\sigma}\right] \\ (\mathcal{Z}_k^{gh})_\nu{}^\mu &= \delta_\nu{}^\mu \ .\end{aligned} \tag{2.41}$$

Here, the symmetric unit tensor and trace free projector in the space of symmetric tensor are written as $\mathbf{1}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2}\left(\delta_\mu{}^\rho \delta_\nu{}^\sigma + \delta_\nu{}^\rho \delta_\mu{}^\sigma\right)$ and $(P_\phi)_{\mu\nu}{}^{\rho\sigma} = \frac{\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma}}{d}$ respectively.

Structure of the β functions and its solution : With the above mentioned cut-off in hand, the operators for $\bar{h}$ and ϕ sector are presented respectively

$$\begin{aligned}\left(\kappa^{-2}\Gamma_k^{(2)}[g,\bar{g}] + \mathcal{R}_k^{grav}\right)_{\bar{h}\bar{h}} &= Z_{Nk}\left[-D^2 + k^2 R^{(0)}\left(-\frac{D^2}{k^2}\right) - 2\Lambda(k) + C_T R\right] \\ \left(\kappa^{-2}\Gamma_k^{(2)}[g,\bar{g}] + \mathcal{R}_k^{grav}\right)_{\phi\phi} &= -\frac{d-2}{2d} Z_{Nk}\left[-D^2 + k^2 R^{(0)}\left(-\frac{D^2}{k^2}\right) - 2\Lambda(k) + C_S R\right]\end{aligned} \tag{2.42}$$

and for ghost sector, the Faddeev-Popov operator is obtained as

$$-M + \mathcal{R}_k^{gh} = -D^2 + k^2 R^{(0)}\left(-\frac{D^2}{k^2}\right) + C_V R \tag{2.43}$$

where $C_V \equiv -\frac{1}{d}$. At this stage, the bars from the metric can be omitted setting $\bar{g} = g$. The heat kernel and Laplace transform techniques are now required for the derivation of the right-hand side of the flow equation eq.(2.26) which comprises of the terms given in eq(s).(2.42, 2.43). Using the heat kernel method which is nicely described in [99], and comparing the coefficients of $\sqrt{g}$ and $\sqrt{g}R$ from both sides of the eq.(2.26), here are the equations which are read off respectively

$$\partial_t \left(Z_{Nk}\Lambda(k)\right) = (16\kappa^2)^{-1}(4\pi)^{d/2}k^d \Big[2d(d+1)\Phi^1_{d/2}\left(-2\Lambda(k)/k^2\right) - 8d\Phi^1_{d/2}(0) \\ -d(d+1)\eta_N\tilde{\Phi}^1_{d/2}\left(-2\Lambda(k)/k^2\right)\Big] \quad (2.44)$$

and

$$\partial_t Z_{Nk} = -(24\kappa^2)^{-1}(4\pi)^{-d/2}k^{d-2}\left[d(d+1)\left\{\Phi^1_{d/2-1}\left(-\frac{2\Lambda(k)}{k^2}\right) - \frac{1}{2}\eta_N\tilde{\Phi}^1_{d/2-1}\left(-2\Lambda(k)/k^2\right)\right\} \\ -6d(d-1)\left\{\Phi^2_{d/2}\left(-2\Lambda(k)/k^2\right) - \frac{1}{2}\eta_N\tilde{\Phi}^2_{d/2}\left(-2\Lambda(k)/k^2\right)\right\} - 4d\Phi^1_{d/2-1}(0) - 24\Phi^2_{d/2}(0)\right] . \quad (2.45)$$

Here, $\eta_N \equiv k\partial_k \ln Z_{Nk} \equiv k\frac{\partial_k G(k)}{G(k)}$ is an anomalous dimension, which will depend on the two known functions of the cosmological constant.

The above two equations give a coupled system of RG equations for the dimensionless running Newton constant $\tilde{g}(k) \equiv k^{d-2}G(k) \equiv k^{d-2}Z^{-1}_{Nk}\bar{G}$ and the dimensionless cosmological constant $\lambda(k) \equiv \Lambda(k)/k^2$

$$k\partial_k\tilde{g} = (d-2+\eta_N)\tilde{g}\,, \quad (2.46)$$

$$k\partial_k\lambda = -(2-\eta_N)\lambda + \frac{1}{2}\tilde{g}(4\pi)^{(1-d/2)}\Big[2d(d+1)\Phi^1_{d/2}(-2\lambda) - 8d\Phi^1_{d/2}(0) \\ -d(d+1)\eta_N\tilde{\Phi}^1_{d/2}(-2\lambda)\Big] . \quad (2.47)$$

These can be thought of as individual flow equations for $\tilde{g}$ and λ, with

$$\eta_N(\tilde{g},\lambda) = \frac{\tilde{g}B_1(\lambda)}{1-\tilde{g}B_2(\lambda)} \quad (2.48)$$

being the anomalous dimension of the operator $\sqrt{g}R$, where the functions $B_1(\lambda)$ and $B_2(\lambda)$ are given by

$$B_1(\lambda) \equiv \frac{1}{3}(4\pi)^{(1-d/2)}\Big[d(d+1)\Phi^1_{d/2-1}(-2\lambda) - 6d(d-1)\Phi^2_{d/2}(-2\lambda) \\ -4d\Phi^1_{d/2-1}(0) - 24\Phi^2_{d/2}(0)\Big]\,, \tag{2.49}$$

$$B_2(\lambda) \equiv -\frac{1}{6}(4\pi)^{(1-d/2)}\Big[d(d+1)\tilde{\Phi}^1_{d/2-1}(-2\lambda) - 6d(d-1)\tilde{\Phi}^2_{d/2}(-2\lambda)\Big]\,. \tag{2.50}$$

The functions $\Phi^p_n(w)$ and $\tilde{\Phi}^p_n(w)$ appearing in these expressions are given by

$$\Phi^p_n(w) = \frac{1}{\Gamma(n)}\int_0^\infty dz z^{n-1}\frac{R^{(0)}(z) - zR^{(0)'}}{[z+R^{(0)}(z)+w]^p}\,, \tag{2.51}$$

$$\tilde{\Phi}^p_n(w) = \frac{1}{\Gamma(n)}\int_0^\infty dz z^{n-1}\frac{R^{(0)}(z)}{[z+R^{(0)}(z)+w]^p}\,. \tag{2.52}$$

The discussion reviewed till now is valid for any dimension d. In the rest of this thesis, we shall work in $d=4$. In that case the flow equations eq.(2.46) and eq.(2.47) take the form

$$k\partial_k\tilde{g} = (2+\eta_N)\tilde{g} \tag{2.53}$$

$$k\partial_k\lambda = -(2-\eta_N)\lambda + \frac{\tilde{g}}{2\pi}\Big[10\Phi^1_2(-2\lambda) - 8\Phi^1_2(0) - 5\eta_N\tilde{\Phi}^1_2(-2\lambda)\Big]\,. \tag{2.54}$$

Recasting the above equations in terms of $G(k)$, $\Lambda(k)$, it can be written

$$k\partial_k G(k) = \eta_N G(k) \tag{2.55}$$

$$k\partial_k\Lambda(k) = \eta_N\Lambda(k) + \frac{1}{2\pi}k^4 G(k)\Big[10\Phi^1_2(-2\Lambda(k)/k^2) - 8\Phi^1_2(0) - 5\eta_N\tilde{\Phi}^1_2(-2\Lambda(k)/k^2)\Big] \tag{2.56}$$

where the functions $B_1(\lambda)$ and $B_2(\lambda)$ in $d=4$ are given by

$$B_1(\lambda) \equiv -\frac{1}{3\pi}\Big[18\Phi^2_2(-2\lambda) - 5\Phi^1_1(-2\lambda) + 4\Phi^1_1(0) + 6\Phi^2_2(0)\Big]\,, \tag{2.57}$$

$$B_2(\lambda) \equiv \frac{1}{6\pi}\Big[18\tilde{\Phi}^2_2(-2\lambda) - 5\tilde{\Phi}^1_1(-2\lambda)\Big]\,. \tag{2.58}$$

Using the expressions for B_1 and B_2, we can expand the anomalous dimension of the operator $\sqrt{g}R$ for $d=4$ in powers of k^2,

$$\eta_N = k^2 G(k) B_1(\Lambda(k)/k^2)\left[1 + k^2 G(k) B_2(\Lambda(k)/k^2) + k^4 G^2(k) B_2^2(\Lambda(k)/k^2) + \cdots\right] . \tag{2.59}$$

eqs.(2.55) and (2.56) cannot be solved exactly. We use an iterative procedure to find the expressions for Λ and G at small k, starting with $\Lambda = 0$ and $\eta_N = 0$. Then both the functions $\Phi_n^p(\Lambda/k^2)$ and $\tilde{\Phi}_n^p(\Lambda/k^2)$ are even functions of k and vanish for $k \to 0$ if $p \geq 1$. It follows that both the functions $B_1(\lambda)$ and $B_2(\lambda)$ and thus also η_N are even functions of k at this order of iteration. Looking at the equations we see that it follows easily from the iterative procedure that both $\Lambda(k)$ and $G(k)$ can be written as power series of only even powers of k,

$$G(k) = G_0\left[1 - \omega G_0 k^2 + \omega_1 G_0^2 k^4 + \mathcal{O}(G_0^3 k^6)\right] \tag{2.60}$$

$$\Lambda(k) = \Lambda_0 + G_0 k^4\left[\nu + \nu_1 G_0 k^2 + \mathcal{O}(G_0^2 k^4)\right] . \tag{2.61}$$

The above expressions look identical to the linearized group flow of λ and $\tilde{g}$ near the trivial fixed point $\lambda = 0$, $\tilde{g} = 0$ [47]. If the couplings are on a generic flow near the trivial fixed point, we will not find a sensible result, as the β-function hits a singularity at $\lambda(k) = \frac{1}{2}$ at a non-zero value of k for $\Lambda_0 > 0$, with η_N diverging at that point. However, we will see later in this thesis that the consistency conditions arising from the dynamics of Bianchi type-I cosmology fixes $\Lambda_0 = 0$, which puts the couplings on a trajectory which hits the trivial fixed point, avoiding the singularity. Hence there is no obstruction to taking the limit $k \to 0$.

In any case, the expressions in eq.(2.60) and eq.(2.61) represent the "quantum corrected" G and Λ. The constants ω, ν, ω_1 and ν_1 can be calculated by inserting these expansions into eq.(2.55) and eq.(2.56),

$$\omega = -\frac{1}{2}B_1(0) = \frac{1}{6\pi}\left[24\Phi_2^2(0) - \Phi_1^1(0)\right] = \frac{4}{\pi}\left(1 - \frac{\pi^2}{144}\right) , \tag{2.62}$$

$$\nu = \frac{1}{4\pi}\Phi_2^1(0) = \frac{\zeta(3)}{2\pi}\,, \tag{2.63}$$

$$\omega_1 = \omega^2 - \frac{B_2(0)}{2}\omega - \frac{13\nu}{6\pi} = \omega^2 - \frac{\omega}{3\pi} - \frac{13\nu}{6\pi}\,, \tag{2.64}$$

$$\nu_1 = -\omega\nu + \frac{5\omega}{6\pi}\tilde{\Phi}_2^1(0) + \frac{5\nu}{3\pi}\Phi_2^2(0) = -\omega\nu + \frac{5\omega}{6\pi} + \frac{5\nu}{3\pi}\,, \tag{2.65}$$

with $B_2(0) = \frac{2}{3\pi}$, $\Phi_2^1(0) = 2\zeta(3)$, $\tilde{\Phi}_2^2(0) = 1$ and $\Phi_2^2(0) = 1$. We remark that these expressions are accurate only if we choose $\Lambda_0 = 0$, as was noted in [26, 51]. We also remark that changing the regulator function $R^{(0)}$ does not change the form of the power series expansions of G and Λ, but will modify the values of the constants above.

2.2.3 Solution of $G(k)$ with vanishing Λ

In this subsection, we will discuss the flow of dimensionless gravitational constant $\tilde{g}(k)$ as the sole coupling constant following literatures [99, 111], omitting the second term, that is, $\Lambda(k)\int d^dx\sqrt{g}$ from the ansatz of the effective action effective action Γ_k in eq.(2.31). The flow equation for $\tilde{g}$ is written removing the flow equation for cosmological constant and letting $\lambda \equiv 0$ in eq.(2.53)

$$\begin{aligned}
\beta_{\tilde{g}}(\tilde{g}(k)) \equiv k\partial_k\tilde{g} &= (2+\eta_N)\tilde{g} \\
&= \left[2 + \frac{\tilde{g}B_1(0)}{1-\tilde{g}B_2(0)}\right]\tilde{g}(k) \\
&= 2\tilde{g}\frac{1-\omega'\tilde{g}}{1-B_2\tilde{g}}\,.
\end{aligned} \tag{2.66}$$

The cut-off dependent constants $B_1(0)$ and $B_2(0)$ for the exponential cut-off given in eq. (2.28) are designated from eqs.(2.57) and (2.58) as

$$\begin{aligned}
B_1(0) &= -\frac{1}{3\pi}\left[24\Phi_2^2(0) - \Phi_1^1(0)\right] \\
B_2(0) &= \frac{1}{6\pi}\left[18\tilde{\Phi}_2^2(0) - 5\tilde{\Phi}_1^1(0)\right]\,.
\end{aligned} \tag{2.67}$$

In equation (2.66), the constants are shortened for ease in the following way:

$$\omega \equiv -\frac{1}{2}B_1(0), \qquad\qquad \omega' \equiv \omega + B_2(0)\ . \tag{2.68}$$

The numerical value of ω, $B_1(0)$ and $B_2(0)$ are given in the above subsection in eq.(2.62). Here, we note that the beta function has two fixed point for two zeros of $\beta_{\tilde{g}}$, namely the Gaussian fixed point (GFP) at $\tilde{g}_*^{GFP} = 0$ and non-Gaussian fixed point at $\tilde{g}_*^{NGFP} = \frac{1}{\omega'}$. Since the beta function for trajectories in between $\tilde{g}(k) \in [0, \tilde{g}_*^{NGFP}]$ is positive, that indicates the gravitational constant is positive, so that we work with these trajectories only. For that, the differential equation (2.66) takes integral form as

$$\int_{\tilde{g}(k_0)}^{\tilde{g}} \frac{d\tilde{g}}{\tilde{g}} - \frac{\omega}{\omega'}\int_{\tilde{g}(k_0)}^{\tilde{g}} \frac{d\tilde{g}}{1-\omega'\tilde{g}} = \int_{k_0}^{k}\frac{dk}{k}\ . \tag{2.69}$$

Integrating the above equation analytically, the solution for the dimensionless gravitational constant can be written as

$$\frac{\tilde{g}}{(1-\omega'\tilde{g})^{\frac{\omega}{\omega'}}} = \frac{\tilde{g}(k_0)}{(1-\omega'\tilde{g}(k_0))^{\frac{\omega}{\omega'}}}\left(\frac{k}{k_0}\right)^2 \tag{2.70}$$

where k_0 is the reference scale. To write the approximate solution for $\tilde{g}(k)$ in the closed form, we observe that the $\frac{\omega}{\omega'}$ is close to unity as $\omega \approx 1.2$, $B_2(0) \approx 0.21$ and $\omega' \approx 1.4$. The close to accurate solution for $\tilde{g}(k)$ is written replacing $\frac{\omega}{\omega'} \to 1$ as

$$\tilde{g}(k) = \frac{\tilde{g}(k_0)k^2}{\omega\tilde{g}(k_0)k^2 + [1-\omega\tilde{g}(k_0)]k_0^2}\ . \tag{2.71}$$

Replacing $\tilde{g}(k)$ with the dimensionful Newton constant as $\tilde{g}(k) = G(k)k^2$, the solution reads

$$G(k) = \frac{G(k_0)}{1+\omega G(k_0)[k^2-k_0^2]}\ . \tag{2.72}$$

Now setting $k_0 = 0$ for the reference scale and identifying $G(0)$ with the observed Newton constant G_0, the running gravitational constant can be written in the simple form with the momentum scale k as

$$G(k) = \frac{G_0}{1 + \omega G_0 k^2} \ . \tag{2.73}$$

This equation provides a smooth picture of crossover between the classical or low energy regime where $G(k) = G_0$ and the ultraviolet regime where $\tilde{g}(k) = \text{const} = \tilde{g}_*^{NGFP}$. When $k \approx m_{Pl}$, or close to the Planck scale indicated by $m_{Pl} \equiv G_0^{-1/2}$, those regimes cross over. This nearly precise $G(k)$ solution will be applied to obtain the quantum corrected Schwarzschild metric [111], which is discussed in chapter 5.

Chapter 3

Bianchi-I cosmology using renormalization group approach for quantum gravity

3.1 Introduction

This chapter is based on the paper [42, 43]. As discussed in the earlier chapter, a natural framework for describing gravity in the setting of quantum field theory is provided by the asymptotic safety (AS) theory for quantum gravity, which is currently addressing one of the most difficult issue in theoretical physics, the creation of a full ultra-violet (UV) theory of gravity. Weinberg has put forth the intriguing proposal that the idea of asymptotic safety may allow a gravitational theory to be non-perturbatively renormalizable [80]. Moreover, the functional renormalization group (FRG) offers a perfect technique for asymptotic safe gravity for studying the phenomenological consequences of quantum gravity since the key object of FRG, the scale dependent effective action interpolates between the short and long-distance regimes smoothly. The functional renormalization group equation (FRGE) governs the evolution of gravitational effective action with the RG energy scale. However, a clearly defined, finite RG trajectory connects the high energy realm and the low energy regime of classical general relativity. Scale-dependence is incorporated into coupling constants by projecting this RG flow from the infinite dimensional space of all functionals into a two-dimensional subspace involving only $\sqrt{g}$ and $\sqrt{g}R$ (Einstein-Hilbert truncation) [26, 31, 47]. There are various implications of AS gravity with this Einstein truncation in cosmologies at late time which can be derived writing Einstein's equation in terms of the “running" Newton's constant and the “running" cosmological constant which are both dependent on the energy scale of the problem [51, 52, 54–57, 61, 62, 112]. In the context of cosmology, this scale may be taken to be a function only of the cosmological time, so the running constants become time-dependent [51, 57, 112].

The Friedmann-Lemaître-Robertson-Walker (FLRW) model of cosmology was studied in [51] using this “RG-improved" Einstein's equation, leading to coupled ordinary differential equations for the scale factor $a(t)$, the density $\rho(t)$, Newton's constant $G(t)$, cosmological constant $\Lambda(t)$. In this chapter, we investigate Bianchi I models of cosmology using RG-improved Einstein's equation, to check if this scheme introduces additional conditions under which the Bianchi I anisotropic cosmology approaches the FLRW universe. We will look at the late time behaviour of the Bianchi-I universe for three different kinds of matter, namely, dust, radiation, and stiff matter. Here, we examine how the Bianchi-I cosmological solution for different matters flows to late times employing the consistency conditions obtained from the renormalization group improved Einstein equations. Now, in order to better comprehend the various difficulties in cosmology, it is important that we briefly cover the fundamentals of cosmology before talking about the Bianchi-I cosmology.

The organization of this chapter is as follows. In Sec. 3.2, we briefly go over cosmological fundamentals and the evolution equation for the FRW universe. In Sec. 3.3 we use the effective average action formalism for gravity which leads to the flow equations for the scale-dependent Newton's constant $G(k)$ and the cosmological constant $\Lambda(k)$ to write the RG improved Einstein's equation for the scale factors. We conclude with a discussion of our results.

3.2 Brief discussion on cosmology

The central theme of cosmology is based on the Copernican principle which states that the universe is pretty much the same everywhere. However, the Copernican principle only applies on the large scales (of 100 Mpc or more), where the local variations in density are averaged out. Such a concept is supported by a wealth of observational data, such as $3K$ cosmic microwave background (CMB) radiation which is nearly complete isotropy. The deviations from regularity in the microwave background radiation are on the order of 10^{-5} or less, which is undoubtedly a sufficient basis for an approximation of spacetime on large scales, even though we now know that it is not perfectly smooth.

The observed universe on scale 100 Mpc or more possess two mathematically precise properties: isotropy and homogeneity which are associated to the Copernican principle. Here, isotropy applies at a specific point which states that the universe looks the same no matter in which direction you point your telescope. The statement that the metric is same throughout the manifold is referred to as homogeneity. To put it another way, the universe has no preferred location regardless of where you set up your telescope. It should be noted that homogeneity and isotropy do not necessarily go hand in hand; a manifold may be homogeneous but not anywhere isotropic or it may be isotropic around a point but not be homogeneous. However, cosmologists have embraced the Copernican principle, which claims that the we are not the centre of the universe and other observers notice isotropy as well, so that homogeneity and isotropy can be taken for granted for large scales.

The usefulness of homogeneity and isotropy is that they imply invariance under rotation and translation respectively, which in turns guarantee the space is maximally symmetric. When homogeneity and isotropy are combined, a space is said to have as many Killing vectors as is physically conceivable. To assert that spacetime itself is maximally symmetric would be an extreme application of the Copernican principle. In reality, it turns out that this isn't the case; based on our knowledge of observation, the universe is isotropic and homogeneous in space, but not in all of spacetime.

In order to represent real world, the "perfect" Copernican principle, which presumes symmetry throughout space and time, must be abandoned. Positing that the universe is spatially isotropic and homogeneous but evolving with time is simple and consistent with observation. In other words, the universe can be foliated into space like slices with each three-dimensional slice being maximally symmetric. So, the spacetime metric has the following form

$$ds^2 = dt^2 + a^2(t)d\mathbf{x}^2 \tag{3.1}$$

where t is the coordinate time, $a(t)$ is known as the scale factor and $\mathbf{x}$ is the 3-vector in space. The scale factor must be independent of the spatial coordinate in order to be homogeneous. This evolution of the universe is characterised by the scale factor. The evolution is measured by the quantity called Hubble parameter $H(t) = \frac{\dot{a}}{a}$. Now for spatial geometry only, the symmetries permit one of the following formats of writing the $d\mathbf{x}$ in spherical coordinates as

$$d\mathbf{x}^2 = \frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{3.2}$$

for $K = \pm 1, 0$ which correspond to spatially spherical (closed universe for $K = 1$), hyperbolic (open universe for $K = -1$) and flat universe (for $K = 0$). However, the spatially flat solution is greatly favoured by the observational constraints, despite the fact that each of the three configurations is a priori conceivable. Unless otherwise indicated, it will always be assumed that $K = 0$ in the following.

Friedmann equation: Now to determine how the universe has evolved, the next step is to include the FLRW metric into Einstein's equation to derive the Friedmann equations connecting the scale factor to the energy-momentum tensor of the universe. Due to the symmetry, the Einstein's equation is reduced to a pair of equations for FLRW metric

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_0}{3}\rho - \frac{K}{a^2} \tag{3.3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_0}{3}\left(\rho + 3p\right) \tag{3.4}$$

taking the energy-momentum tensor as a perfect fluid. The energy-momentum tensor of a perfect fluid represents the cosmic matter. This is of the form

$$T_{\mu\nu} = (p+\rho)v_\mu v_\nu + pg_{\mu\nu} \tag{3.5}$$

where p is the pressure, ρ is the energy density and v_μ is the four velocity of the fluid which satisfies the relation $v^\mu v_\mu = -1$. It can always be changed to the form because of the imposed homogeneity and isotropy $T^\mu{}_\nu = \mathrm{diag}(-\rho, p, p, p)$.
A third equation can be derived considering the covariant conservation of the energy-momentum tensor, that is, $\nabla_\mu T^{\mu\nu} = 0$

$$\dot{\rho} + 3H\left(p + \rho\right) = 0 \tag{3.6}$$

which is called the continuity equation.
Now, one can see that there are two independent equations among the three, namely, the first Friedmann equation, the second Friedmann equation, also known as the acceleration equation, and the continuity equation. Thus we have a system of two independent equations with three unknowns $a(t)$, ρ and p. To make progress, the equation of state can be chosen relating the pressure of the perfect fluid with the matter density of the fluid

$$p = \Omega\rho \tag{3.7}$$

where Ω is a constant and independent of time. To obtain $a(t)$, ρ and p for all times, we can now solve eq.(s)(3.3, 3.6 and 3.7). However, the fact that our universe is made up of different components with different equations of state really complicates how our universe evolved. Fortunately for the cause of simplicity, the pressure and the energy density for the various matter components are additive. Since there is no interaction between the components, the continuity equation holds for each one separately. We will investigate the cosmology for a single component throughout the thesis because different components evolve at different rates, so that, one type of source will be clearly dominant by one component for longer periods of time. Using eq.(3.7) in the continuity equation, the matter density falls off as

$$\rho = \rho_0 a^{-3(1+\Omega)} \ . \tag{3.8}$$

where ρ_0 is the matter density at present time.

Evolution of the scale factor : For spatially flat, single component universe, the first Friedmann equation becomes

$$\dot{a}^2 = \frac{8\pi G_0 \rho_0}{3} a^{-(1+3\Omega)} \ . \tag{3.9}$$

This can be immediately integrated to acquire the scale factor with the restriction $\Omega \neq 1$

$$a(t) = \left[1 + \frac{3(1+\Omega)}{2}\sqrt{\frac{8\pi G_0 \rho_0}{3}}(t - t_0)\right]^{\frac{2}{3+3\Omega}} \tag{3.10}$$

where the initial condition is set taking $a(t = t_0) = 1$, where t_0 being the present time of the universe. So the Hubble parameter for spatially flat, single component universe at any time is given by

$$H = \left(\frac{\dot{a}}{a}\right)_t = \frac{2}{3(1+\Omega)} t^{-1} \ . \tag{3.11}$$

As a function of cosmological time, the energy density of the universe for a component having the equation of state parameter Ω is given by

$$\rho(t) = \rho_0 \left[1 + \frac{3(1+\Omega)}{2} \sqrt{\frac{8\pi G_0 \rho_0}{3}} (t - t_0) \ . \right]^{-2} \tag{3.12}$$

The energy density changes with cosmological time as t^{-2} irrespective of the value of Ω. The fundamental discussion of cosmology is useful for comprehending cosmology while taking into account quantum corrections to the gravitational constant and cosmological constant.

3.3 Bianchi I universe with running G and Λ

In this section, we are interested in the anisotropic cosmology described by Bianchi I metric

$$ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2 \, . \tag{3.13}$$

For time varying G and Λ this cosmological model has been studied in the presence of a perfect fluid [113–116]. Time variation of G and Λ have also been considered in flat FLRW cosmological models [117].

For cosmology we will need to consider a scale defined by the cosmological time t, so we first improve Einstein's field equations of general relativity with time-varying Newton's gravitational constant G and cosmological constant Λ

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G(t) T_{\mu\nu} - \Lambda(t) g_{\mu\nu} \ . \tag{3.14}$$

Here $G(t)$ and $\Lambda(t)$ are related to the scale dependent $G(k)$ and $\Lambda(k)$ and take into account the leading quantum corrections coming from the renormalization group flow. Note that the RG improvement has been made at the level of the field equations and is a kind of “shortcut" which captures the effects of quantum gravity to leading order. For the metric eq.(3.13) together with the energy-momentum tensor eq.(3.5), we obtain the usual equations for the scale factors

$$-\frac{\ddot{b}}{b}-\frac{\ddot{c}}{c}-\frac{\dot{b}\dot{c}}{bc}=8\pi Gp-\Lambda \tag{3.15}$$

$$-\frac{\ddot{a}}{a}-\frac{\ddot{c}}{c}-\frac{\dot{a}\dot{c}}{ac}=8\pi Gp-\Lambda \tag{3.16}$$

$$-\frac{\ddot{a}}{a}-\frac{\ddot{b}}{b}-\frac{\dot{a}\dot{b}}{ab}=8\pi Gp-\Lambda \tag{3.17}$$

$$\frac{\dot{a}\dot{b}}{ab}+\frac{\dot{b}\dot{c}}{bc}+\frac{\dot{a}\dot{c}}{ac}=8\pi G\rho+\Lambda\,. \tag{3.18}$$

Further, the covariant conservation of the energy-momentum tensor $\nabla_\mu T^{\mu\nu}=0$ for Bianchi-I universe yields

$$\dot{\rho}+(p+\rho)\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c}\right)=0\,. \tag{3.19}$$

Now since the Einstein tensor is covariantly conserved, the right hand side of eq.(3.14) must also be covariantly conserved. This leads to the consistency equation

$$8\pi\rho\dot{G}+\dot{\Lambda}=0\,, \tag{3.20}$$

where the dot denotes time derivative.

Two of the equations for scale factors can be combined to produce

$$\frac{d}{dt}\left[\ln\left(\frac{\dot{a}}{a}-\frac{\dot{b}}{b}\right)\right]+\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c}\right)=0\,, \tag{3.21}$$

integrating which we get

$$\frac{\dot{a}}{a}-\frac{\dot{b}}{b}=\frac{k_1}{\mathcal{R}^3(t)}\,, \tag{3.22}$$

where k_1 is a constant of integration and $\mathcal{R}^3(t) = abc$. A similar calculation using the other pairs yields

$$\frac{\dot{b}}{b} - \frac{\dot{c}}{c} = \frac{k_2}{\mathcal{R}^3(t)} \tag{3.23}$$

$$\frac{\dot{c}}{c} - \frac{\dot{a}}{a} = \frac{k_3}{\mathcal{R}^3(t)} \tag{3.24}$$

where k_2 and k_3 are integration constants, satisfying $k_1 + k_2 + k_3 = 0$. Let us rename for convenience the integration constants as l, $l\beta$ and $-l(1+\beta)$. Integrating these equations we get

$$a(t) = m_1 \mathcal{R}(t) \exp\left[\frac{l(2+\beta)}{3} \int \frac{dt}{\mathcal{R}^3(t)}\right] \tag{3.25}$$

$$b(t) = m_2 \mathcal{R}(t) \exp\left[\frac{l(\beta-1)}{3} \int \frac{dt}{\mathcal{R}^3(t)}\right] \tag{3.26}$$

$$c(t) = m_3 \mathcal{R}(t) \exp\left[-\frac{l(1+2\beta)}{3} \int \frac{dt}{\mathcal{R}^3(t)}\right], \tag{3.27}$$

where m_1, m_2, m_3 are arbitrary constants of integration satisfying $m_1 m_2 m_3 = 1$.

In this setup, we wish to consider the late time effect of quantum gravity. Let us use the long distance perturbative series expansion of $G(k)$ and $\Lambda(k)$ of eq.(2.60) and eq.(2.61) from chapter 2, suitably converted to a time-varying form. The identification of the infrared cutoff for momentum scale k involves expressing k in terms of all scales that are relevant to the problem under consideration. In the case of the FLRW universe, homogeneity and isotropy of spacetime imply that k is a function of the cosmological time only. Hence the constants G and Λ take the form

$$G(t) \equiv G(k = k(t)), \qquad \Lambda(t) \equiv \Lambda(k = k(t)) \,. \tag{3.28}$$

For the anisotropic Bianchi type-I spacetime, we still have homogeneity so that all scale factors are functions of cosmological time only. Let us then consider the theory at a momentum scale set by the cosmological time, $k \equiv k(t)$ in this case also. As pointed out in [51], there are two natural choices of scale in an FLRW universe that could relate t

to k. One is where $k \sim t^{-1}$, which is to say that the theory is cut off at a wavelength determined by how far signals can have traveled during the lifetime of the universe, disregarding the expansion of the universe. The other choice is to include some effect of expansion by choosing $k \sim \mathcal{R}^{-1}$. In this case, the equations have no consistent solution with this choice for the FLRW universe with ordinary matter. For exotic matter there is a consistent solution for the choice of $k \propto \mathcal{R}(t)^{-1}$. As we will see below, for a Bianchi I universe we need to include higher order terms in t^{-1}. Then the simplest such behaviour at late times is

$$k = \sum_n \frac{\xi_n}{t^n}\,. \tag{3.29}$$

We also note that a choice of different cutoff scales in different directions does not seem practicable.

For our analysis, we will keep terms up to $n = 3$ in eq.(3.29) because we are interested in the behaviour of G and Λ up to $\mathcal{O}\left(\frac{t_{Pl}^4}{t^4}\right)$, so we will employ the cutoff

$$k = \frac{\xi}{t} + \frac{\sigma}{t^2} + \frac{\delta}{t^3}\,. \tag{3.30}$$

Inserting this expression for k into the series for $G(k)$ and $\Lambda(k)$, we obtain the time dependent Newton's gravitational constant and cosmological constant in the perturbative or low energy regime,

$$G(t) = G_0\left[1 - \frac{\tilde{\omega}G_0}{t^2}\left(1 + \frac{2\tilde{\sigma}}{t} + \frac{2\tilde{\delta}}{t^2} + \frac{\tilde{\sigma}^2}{t^2}\right) + \frac{\tilde{\omega_1}G_0^2}{t^4} + \mathcal{O}\left(\frac{t_{Pl}^6}{t^6}\right)\right], \tag{3.31}$$

$$\Lambda(t) = \Lambda_0 + \frac{G_0}{t^4}\left[\tilde{\nu}\left(1 + \frac{4\tilde{\sigma}}{t} + \frac{4\tilde{\delta}}{t^2} + \frac{6\tilde{\sigma}^2}{t^2}\right) + \frac{\tilde{\nu_1}G_0}{t^2} + \mathcal{O}\left(\frac{t_{Pl}^4}{t^4}\right)\right], \tag{3.32}$$

where we have defined $\tilde{\omega} \equiv \omega\xi^2$, $\tilde{\omega_1} \equiv \omega_1\xi^4$, $\tilde{\nu} \equiv \nu\xi^4$, $\tilde{\nu_1} \equiv \nu_1\xi^6$, $\tilde{\sigma} \equiv \frac{\sigma}{\xi}$ and $\tilde{\delta} \equiv \frac{\delta}{\xi}$ for convenience.

To proceed further we now assume, in line with the arguments of [51], that renormalization effects coming from the matter sector are small compared to those of pure quantum gravity. Then the equation of state relating the pressure p and the energy density ρ is linear, is assumed to be $p(t) = \Omega\rho(t)$, as taken in equation (3.7). We next substitute the equation of state into the energy-momentum conservation law eq.(3.19) and integrate it. This gives

$$\rho\,[\mathcal{R}(t)]^{3(1+\Omega)} = \frac{\mathcal{M}}{8\pi}\,, \tag{3.33}$$

where $\mathcal{M}$ is an integration constant. On the other hand, using eq.(3.20) we can express the energy density $\rho(t)$ in the form

$$\rho = -\frac{1}{8\pi}\frac{\dot{\Lambda}}{\dot{G}}\,. \tag{3.34}$$

Combining eq.(3.33) and eq.(3.34), we obtain

$$\mathcal{R}(t) = \left[-\frac{\mathcal{M}\dot{G}}{\dot{\Lambda}}\right]^{\frac{1}{3+3\Omega}} \quad . \tag{3.35}$$

The time derivatives of $G(t)$ and $\Lambda(t)$ can be calculated from their expressions above and using them in the expressions for the energy density ρ and the average scale factor $\mathcal{R}$, we find

$$\rho(t) = \frac{1}{4\pi}\left(\frac{\tilde{\nu}}{\tilde{\omega}}\right)\frac{1}{G_0 t^2}\left\{1 + \frac{2\tilde{\sigma}}{t} + \frac{2\tilde{\delta}}{t^2} + \frac{\tilde{\sigma}^2}{t^2} + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right)\frac{G_0}{t^2} + \mathcal{O}\left(\frac{G_0^2}{t^4}\right)\right\}\,, \tag{3.36}$$

$$\mathcal{R}(t) = \left[\frac{\mathcal{M}G_0}{2}\left(\frac{\tilde{\omega}}{\tilde{\nu}}\right)\right]^{\frac{1}{(3+3\Omega)}} t^{\frac{2}{(3+3\Omega)}}\left\{1 - \frac{1}{(3+3\Omega)}\left(\frac{2\tilde{\sigma}}{t} + \frac{2\tilde{\delta}}{t^2} - \frac{(3\Omega+5)}{(3+3\Omega)}\frac{\tilde{\sigma}^2}{t^2} + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right)\frac{G_0}{t^2}\right) + \mathcal{O}\left(\frac{G_0^2}{t^4}\right)\right\}. \tag{3.37}$$

We now discuss three cases of cosmic matter separately: i) dust, for which $\Omega = 0$; ii) radiation, $\Omega = \frac{1}{3}$; and iii) stiff fluid, $\Omega = 1$. The first two cases have something in common, as we will see now. For $\Omega \neq 1$ and defining $\alpha = \left[\frac{\mathcal{M}G_0}{2}\left(\frac{\tilde{\omega}}{\tilde{\nu}}\right)\right]^{-\frac{1}{1+\Omega}}$, we can integrate the expressions for the scale factors to find

$$\begin{aligned} a(t) &= m_1\mathcal{R}(t)\exp\left[\frac{l(2+\beta)\alpha}{3}\mathcal{N}(t)\right]\,, \\ b(t) &= m_2\mathcal{R}(t)\exp\left[\frac{l(\beta-1)\alpha}{3}\mathcal{N}(t)\right]\,, \\ c(t) &= m_3\mathcal{R}(t)\exp\left[-\frac{l(1+2\beta)\alpha}{3}\mathcal{N}(t)\right]\,, \end{aligned} \tag{3.38}$$

where we have written

$$\begin{aligned} \mathcal{N}(t) &= \int \frac{dt}{\mathcal{R}^3(t)} \\ &= \frac{(\Omega+1)}{(\Omega-1)}t^{\frac{(\Omega-1)}{(\Omega+1)}} - \tilde{\sigma}t^{-\frac{2}{(\Omega+1)}} - \frac{1}{(\Omega+3)}\left(2\tilde{\delta} - \frac{(\Omega-1)\tilde{\sigma}^2}{(\Omega+1)} + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right)G_0\right)t^{-\frac{(\Omega+3)}{(\Omega+1)}}\,, \end{aligned} \tag{3.39}$$

and we have neglected higher order terms. From these equations for the scale factors, we can compute the directional Hubble parameters,

$$\begin{aligned} \frac{\dot{a}}{a} &= H(t) + \frac{l(2+\beta)\alpha}{3}H_1(t)\,, \\ \frac{\dot{b}}{b} &= H(t) + \frac{l(\beta-1)\alpha}{3}H_1(t)\,, \\ \frac{\dot{c}}{c} &= H(t) - \frac{l(1+2\beta)\alpha}{3}H_1(t)\,. \end{aligned} \tag{3.40}$$

Here the “average Hubble parameter” $H(t)$ includes isotropic quantum corrections,

$$H(t) = \frac{\dot{\mathcal{R}}}{\mathcal{R}} = \frac{2}{(3+3\Omega)}\frac{1}{t}\left[1 + \frac{\tilde{\sigma}}{t} + \left(2\tilde{\delta} - \tilde{\sigma}^2 + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right)G_0\right)\frac{1}{t^2} + \mathcal{O}\left(\frac{t_{Pl}^3}{t^3}\right)\right]\,, \tag{3.41}$$

while the effects of anisotropy are included in the coefficients of $H_1(t)$, which also includes quantum corrections,

$$H_1(t) = t^{-\frac{2}{1+\Omega}} + \frac{2\tilde{\sigma}}{(1+\Omega)}t^{-\frac{(\Omega+3)}{(1+\Omega)}} + \frac{1}{(\Omega+1)}\left(2\tilde{\delta} - \frac{(\Omega-1)\tilde{\sigma}^2}{(\Omega+1)} + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right)G_0\right)t^{-\frac{2(2+\Omega)}{1+\Omega}}\,. \tag{3.42}$$

We now run a consistency check on the solutions for the scale factors, by putting these solutions into eq.(3.18). Keeping up to $\mathcal{O}\left(\frac{t_{Pl}}{t}\right)^4$, we find

$$3\left(\frac{\dot{\mathcal{R}}}{\mathcal{R}}\right)^2 - \frac{\alpha^2 l^2(\beta^2+\beta+1)}{3} t^{-\frac{4}{1+\Omega}} \left\{1 + \frac{4\tilde{\sigma}}{(1+\Omega)}\frac{1}{t} + \frac{2}{(1+\Omega)}\left(2\tilde{\delta} + \frac{(3-\Omega)}{(1+\Omega)}\tilde{\sigma}^2 + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right)G_0\right)\frac{1}{t^2}\right\}$$
$$= \Lambda_0 + 2(\tfrac{\tilde{\nu}}{\tilde{\omega}})\frac{1}{t^2} + 4(\tfrac{\tilde{\nu}}{\tilde{\omega}})\frac{\tilde{\sigma}}{t^3} - \tilde{\nu}\frac{G_0}{t^4} + 2(\tfrac{\tilde{\nu}}{\tilde{\omega}})\left(2\tilde{\delta} + \tilde{\sigma}^2 + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right)G_0\right)\frac{1}{t^4}\,. \quad (3.43)$$

The above equation is the consistency relation which the scale factor must satisfy for all $\Omega \neq 1$. Let us now see the consequences of the above relation for $\Omega = 0$ (dust) and $\Omega = \frac{1}{3}$ (radiation) .

3.3.1 $\Omega = 0$

Comparing the coefficients of different powers of t on both sides of eq.(3.43) we get the following consistency conditions

$$\Lambda_0 = 0\,, \quad (3.44)$$
$$\frac{\tilde{\omega}}{\tilde{\nu}} = \frac{3}{2}\,, \quad (3.45)$$

$$\frac{4}{3}\left[4\tilde{\delta} - \tilde{\sigma}^2 + 2\left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right)G_0\right] - \frac{l^2(\beta^2+\beta+1)\alpha^2}{3} = -\tilde{\nu}G_0 + 2\left(\frac{\tilde{\nu}}{\tilde{\omega}}\right)\left[2\tilde{\delta} + \tilde{\sigma}^2 + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right)G_0\right]\,, \quad (3.46)$$

with $\alpha = \left[\frac{\mathcal{M}G_0}{2}\left(\frac{\tilde{\omega}}{\tilde{\nu}}\right)\right]^{-1}$ for $\Omega = 0$.
Combining eq.(3.45) and eq.(3.46), we get a consistency condition valid up to $\mathcal{O}\left(\frac{1}{t^4}\right)$ in our calculations,

$$\frac{8}{3}\tilde{\sigma}^2 - \frac{8}{3}\tilde{\delta} = \tilde{\nu}G_0 + \frac{4G_0}{3}\left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right) - \frac{l^2\alpha^2\left(\beta^2+\beta+1\right)}{3}\,. \quad (3.47)$$

Note that if we had not included the $\mathcal{O}\left(\frac{1}{t^4}\right)$ terms in eq.(3.43), we would have obtained conditions corresponding to FLRW cosmology, which were found in [51]. If we keep terms up to $\mathcal{O}\left(\frac{1}{t^4}\right)$ and compare coefficients, we can immediately conclude that for $l = 0$ we will regain the FLRW universe. Thus we see that for $\Omega = 0$, the anisotropic Bianchi-I cosmology does not necessarily flow to the FLRW solution when quantum corrections are included.

3.3.2 $\Omega = \frac{1}{3}$

In this case, by comparing inverse powers of t in the consistency condition eq.(3.43), we again find $\Lambda_0 = 0$, and

$$\frac{\tilde{\omega}}{\tilde{\nu}} = \frac{8}{3}, \tag{3.48}$$

$$4\left(\frac{\tilde{\nu}}{\tilde{\omega}}\right)\tilde{\sigma} = \frac{3}{2}\tilde{\sigma} - \frac{l^2\alpha^2\left(\beta^2+\beta+1\right)}{3}, \tag{3.49}$$

from which it immediately follows that

$$l^2\alpha^2\left(\beta^2+\beta+1\right) = 0\,. \tag{3.50}$$

If $l \neq 0$, we must have $\beta^2+\beta+1 = 0$. This implies that the two roots of β are complex. Since the scale factors must be real, it follows that $l = 0$. Then from the terms of order t^{-4} in eq.(3.43) we get the condition

$$\frac{3}{2}\left(\tilde{\sigma}^2 - \tilde{\delta}\right) = \tilde{\nu}G_0 + \frac{3}{4}\left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right)G_0\,. \tag{3.51}$$

Hence we see that all the directional Hubble parameters must be equal, i.e. the universe must be FLRW, in the presence of radiation. It is also not difficult to see that for $0 < \Omega < 1$ eq.(3.50) will always appear as a consistency condition, so $l = 0$ and the universe becomes FLRW at late times. Thus we can conclude from the above analysis that the scale factors of anisotropic Bianchi type-I metric flow to the isotropic FLRW cosmology due to renormalization group flow of the Newton's gravitational constant $G(t)$ and the cosmological constant $\Lambda(t)$, for all $0 < \Omega < 1$.

3.3.3 $\Omega = 1$

The case of $\Omega = 1$, which corresponds to stiff matter, is somewhat different. First we write down the expression for the average of the scale factor $\mathcal{R}$ by setting $\Omega = 1$ in eq.(3.37). This produces

$$\mathcal{R}(t) = \left[\frac{\mathcal{M}G_0}{2}\left(\frac{\tilde{\omega}}{\tilde{\nu}}\right)\right]^{\frac{1}{6}} t^{\frac{1}{3}} \left\{1 - \frac{1}{6}\left(\frac{2\tilde{\sigma}}{t} + \frac{2\tilde{\delta}}{t^2} - \frac{4}{3}\frac{\tilde{\sigma}^2}{t^2} + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right)\frac{G_0}{t^2}\right) + \mathcal{O}\left(\frac{t_{Pl}^3}{t^3}\right)\right\} . \tag{3.52}$$

The solutions for the scale factors for stiff matter are then

$$\begin{aligned} a(t) &= m_1 \mathcal{R}(t)\, t^{\frac{l(2+\beta)\alpha}{3}} \exp\left[-\frac{l(2+\beta)\alpha}{3}\mathcal{Q}(t)\right] \\ b(t) &= m_2 \mathcal{R}(t)\, t^{\frac{l(\beta-1)\alpha}{3}} \exp\left[-\frac{l(\beta-1)\alpha}{3}\mathcal{Q}(t)\right] \\ c(t) &= m_3 \mathcal{R}(t)\, t^{-\frac{l(1+2\beta)\alpha}{3}} \exp\left[\frac{l(1+2\beta)\alpha}{3}\mathcal{Q}(t)\right] , \end{aligned} \tag{3.53}$$

where we now have $\alpha = \left[\frac{\mathcal{M}G_0}{2}\left(\frac{\tilde{\omega}}{\tilde{\nu}}\right)\right]^{-\frac{1}{2}}$ for $\Omega = 1$ and

$$\mathcal{Q}(t) = \tilde{\sigma} t^{-1} + \frac{1}{4}\left(2\tilde{\delta} + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right) G_0\right) t^{-2} . \tag{3.54}$$

The directional Hubble parameters are now computed up to $\mathcal{O}\left((\frac{t_{Pl}}{t})^3\right)$,

$$\begin{aligned} \frac{\dot{a}}{a} &= \frac{\dot{\mathcal{R}}}{\mathcal{R}} + \frac{l(2+\beta)\alpha}{3}\bar{H}(t) \\ \frac{\dot{b}}{b} &= \frac{\dot{\mathcal{R}}}{\mathcal{R}} + \frac{l(\beta-1)\alpha}{3}\bar{H}(t) \\ \frac{\dot{c}}{c} &= \frac{\dot{\mathcal{R}}}{\mathcal{R}} - \frac{l(1+2\beta)\alpha}{3}\bar{H}(t) , \end{aligned} \tag{3.55}$$

The (isotropic) average Hubble parameter is

$$\frac{\dot{\mathcal{R}}}{\mathcal{R}} = \frac{1}{3t}\left[1 + \frac{\tilde{\sigma}}{t} + \left(2\tilde{\delta} - \tilde{\sigma}^2 + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right) G_0\right)\frac{1}{t^2} + \mathcal{O}\left(\frac{t_{Pl}^3}{t^3}\right)\right] \tag{3.56}$$

and we have also written, to the same order of approximation,

$$\bar{H}(t) = \frac{1}{t} + \frac{\tilde{\sigma}}{t^2} + \frac{1}{2}\left(2\tilde{\delta} + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right) G_0\right)\frac{1}{t^3} \ . \tag{3.57}$$

As before, we now put these solutions into eq.(3.18) for a consistency check. The consistency condition is then found to be

$$3\left(\frac{\dot{\mathcal{R}}}{\mathcal{R}}\right)^2 - \frac{\alpha^2 l^2(\beta^2+\beta+1)}{3}\left\{\frac{1}{t^2} + \frac{2\tilde{\sigma}}{t^3} + \left(2\tilde{\delta} + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right) G_0\right)\frac{1}{t^4} + \frac{\tilde{\sigma}^2}{t^4}\right\}$$
$$= \Lambda_0 + \frac{2\tilde{\nu}}{\tilde{\omega} t^2} + \frac{4\tilde{\nu}\tilde{\sigma}}{\tilde{\omega} t^3} - \tilde{\nu}\frac{G_0}{t^4} + \frac{2\tilde{\nu}}{\tilde{\omega}}\left(2\tilde{\delta} + \tilde{\sigma}^2 + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right) G_0\right)\frac{1}{t^4} \ . \tag{3.58}$$

Calculating $\left(\frac{\dot{\mathcal{R}}}{\mathcal{R}}\right)^2$ from eq.(3.56) and comparing the coefficients of the $t^0\,, t^{-2}\,, t^{-4}$ terms respectively, we find the equations

$$\Lambda_0 = 0\,, \tag{3.59}$$

$$2\left(\frac{\tilde{\nu}}{\tilde{\omega}}\right) = \frac{1}{3}\left(1 - \alpha^2 l^2(\beta^2+\beta+1)\right)\,, \tag{3.60}$$

$$\frac{2}{3}\left\{2\tilde{\delta} - \frac{\tilde{\sigma}^2}{2} + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right) G_0\right\} - \frac{l^2\alpha^2(\beta^2+\beta+1)}{3}\left\{2\tilde{\delta} + \tilde{\sigma}^2 + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right) G_0\right\}$$
$$= -\tilde{\nu} G_0 + \frac{2\tilde{\nu}}{\tilde{\omega}}\left(2\tilde{\delta} + \tilde{\sigma}^2 + \left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right) G_0\right) \ . \tag{3.61}$$

Using eq.(3.60) in eq.(3.61), we obtain

$$\frac{2}{3}\left(\tilde{\sigma}^2 - \tilde{\delta}\right) = \tilde{\nu} G_0 + \frac{1}{3}\left(\frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}}\right) G_0 \ . \tag{3.62}$$

We note that from consistency condition in eq.(3.60), we can write l in terms of other constant β for $\Omega = 1\,$,

$$l = \frac{1}{\alpha}\frac{\sqrt{1 - 6(\frac{\tilde{\nu}}{\tilde{\omega}})}}{\sqrt{(\beta^2+\beta+1)}} \ . \tag{3.63}$$

As $\frac{\dot{a}}{a}$, $\frac{\dot{b}}{b}$ and $\frac{\dot{c}}{c}$ are real, we get the following condition from the above equation

$$1 - 6\left(\frac{\tilde{\nu}}{\tilde{\omega}}\right) \geq 0 \qquad \Rightarrow \qquad \xi^2 \leq \frac{1}{6}\frac{\omega}{\nu}\,. \tag{3.64}$$

Note that the inequality is saturated for the FLRW case, since from eq.(3.60) we see that $\frac{\tilde{\nu}}{\tilde{\omega}} = \frac{1}{6}$ when $l^2(\beta^2 + \beta + 1) = 0$, which implies that $l = 0$ since β must be real.
Finally, by putting eq.(3.63) into eq.(3.55) we observe that for large β, we get a Kasner type solution, i.e. there are expanding and contracting directions. For large positive β the expanding directions would involve the scale factors a, b and contracting direction would involve c. We further find that the range of β which result in a Kasner type solution change when we take into account the quantum gravity corrections. Likewise we get a Kasner solution for some values of negative β as well.

3.4 Conclusions

In this chapter, we have studied the anisotropic Bianchi-I cosmological model taking quantum gravitational effects into account. The analysis is valid for late times which correspond to the perturbative regime of the exact renormalization group flow of the effective average action for quantum gravity. We used the renormalization group improved cosmological evolution equation by including the scale dependence of Newton's constant and the cosmological constant. We have obtained the solution of G and Λ in power series of the infrared cutoff scale from the cosmological evolution equation.
In an improvement over previous works, we have included higher powers of $1/t$ in the expression for the infrared cutoff scale k. If we had not done this, the consistency condition eq.(3.43) would not hold for any value of $\Omega < 1$. Indeed, because we have considered higher order terms in the expansion of G and Λ in terms of powers of the infrared cutoff, the consistency conditions would fail even for isotropic cosmologies without the additional terms in the expression for k. From this we have found the solution of the energy density and average scale factor in an inverse power series of the cosmological time. An important point to note in this regard is that the power series expansion

of the dimensionful cosmological constant Λ makes sense only if the leading term Λ_0 vanishes, because otherwise the dimensionless λ diverges as $k \to 0$. But if $\Lambda_0 = 0$, the dimensionless couplings $\tilde{g}$ and λ flow on a trajectory directed towards the trivial fixed point $\tilde{g} = 0\,, \lambda = 0$ as the infrared cutoff scale goes to zero. Thus it comes as no surprise that the consistency of Einstein equations with renormalization group flow analysis of $\tilde{g}$ and λ implies that the only allowed value of Λ_0 is zero, for all types of fluids, namely, dust, radiation and stiff matter.

Using these solutions for G and Λ, we have then showed how the flow of anisotropic Bianchi-I cosmology gets affected by quantum gravitational effects for known matter like the dust, radiation and stiff matter. We have computed the scale factors from Einstein equations for dust, radiation and stiff matter for Bianchi-I metric. The consistency conditions following from Einstein equations indicate that the Bianchi-I anisotropic cosmological universe eventually evolves into a FLRW universe at late times if filled with a perfect fluid with the equation of state $p = \Omega\rho$ for $0 < \Omega < 1$. This includes the case of radiation. The scale factors $a(t)\,, b(t)$ and $c(t)$ take the same form and expand in the same rate in all directions. For the $\Omega = 0$ case which corresponds to dust, we find that the Bianchi-I universe does not necessarily flow to the FLRW isotropic universe. For $\Omega = 1$ which corresponds to stiff matter, we observe from the consistency conditions that the solution does not flow to the isotropic FLRW universe at late times. We also calculate a bound on the cutoff parameter ξ and find that the Bianchi-I universe flows to the isotropic FLRW univese at late times if ξ^2 equals its maximum value, but not otherwise. We also find that there is a possibility of getting a Kasner like solution in this case.

Chapter 4

Cosmology with modified continuity equation in asymptotically safe gravity

4.1 Introduction

A consistent quantization of general relativity in four dimensions is the holy grail of theoretical high-energy and gravitational physics. As mentioned earlier, perturbative renormalization of Einstein gravity fails because of the negative mass dimension of the gravitational coupling. An alternative view is that general relativity cannot be quantized directly, but emerges as an effective low energy theory from a quantum action which in principle includes all diffeomorphism-invariant localfunctions of the metric not ruled out by other symmetries [23]. Over the last couple of decades asymptotically safe quantum gravity employing the functional renormalization group (RG) approach, which is consistent across all energy scales, has gained popularity. Since a UV fixed point exists, the theory is asymptotically secure [30–34, 118, 119].

Here, the formulation of asymptotically safe gravity is based on an (Euclidean) "effective average action" $\Gamma_k[g_{\mu\nu}]$ [26, 29, 31, 32, 47, 95, 99, 105], which is detailed in chapter 2. It is defined such that it correctly describes all gravitational phenomena, taking into account the effect of all loops, at a momentum scale k. A scale-dependent RG equation is derived by including all possible diffeomorphism invariant local functions of the metric [26, 33, 47–49]. This results in a running Newton's constant as well as running cosmological constant which reads

$$G(k) = G_0 \left[1 - \omega G_0 k^2 + \mathcal{O}(G_0^2 k^4)\right] , \tag{4.1}$$

$$\Lambda(k) = \nu G_0 k^4 + \mathcal{O}(G_0^2 k^6) \tag{4.2}$$

where G_0 is the Newton's gravitational constant at $k = 0$, and the constants ω and ν are given in eqs. (2.62) and (2.63). For the purpose of investigating cosmic solutions taking the leading order corrections into account, take note that we have only considered the leading order correction from eqs.(2.60) and (2.61). Additionally, as was covered in chapter 2, we start by assuming that $\Lambda_0 = 0$. The constants appearing in Einstein's field equation are then replaced by the running constants to get the "RG-improved" Einstein equation.

The Friedmann-Lemaître-Robertson-Walker (FLRW) model of cosmology was investigated in [51, 53–58, 62, 120–124] using the "RG improved" Einstein equation. Bianchi I cosmology was also discussed in previous chapter following [42] in the same approach. The procedure leads to a set of ordinary coupled differential equations involving the scale factor, density, Newton's universal gravitational constant, cosmological constant, and a cutoff function. An important input that goes into the derivation of the coupled differential equations for the scale factor and density is the covariant conservation of the energy-momentum tensor. Since the Einstein tensor is divergence-free as a geometrical identity, the conservation of the energy-momentum tensor leads to an additional equation involving the time derivatives of the Newton's constant and the cosmological constant. This leads to a consistency equation when we assume an equation of state for the matter [51, 62, 125–128]; once the equation of state is used, we find a system of three equations and two unknowns. The solution of the two unknowns from two of the equations would necessarily need to satisfy the third equation for consistency of the solution. If the matter energy-momentum tensor corresponds to a classical perfect fluid, one may be able to justify the use of the usual conservation equation and thus of the consistency condition. If however there is an underlying quantum theory of the matter, we should in principle include the RG flow of that as well since in general it would contribute to the running of the Newton's gravitational constant and the cosmological constant, and also lead to running masses and coupling constants for themselves [129, 130]. For example, in case of electromagnetic radiation appearing as a source in the RG improved Einstein equations, if its RG flow is considered, then it would result in the running fine structure constant appearing in Einstein equation. In that case it turns out that if we demand that the energy-momentum tensor remain divergence-free, the resulting consistency equation cannot be satisfied at late times. Therefore we should demand conservation, not of the usual $T_{\mu\nu}$, but of the entire right hand side of Einstein equation including the running gravitational and other coupling constants as well as the running cosmological constant. We call this the "modified conservation equation" and apply it to the FLRW cosmology in this paper [55, 60, 62, 125, 126, 131, 132]. We will consider only perfect fluid here as we are mainly interested in finding out how the resulting cosmology differs at late times from the approach of treating the consistency equation separately; the problem of including radiation, and thus the running fine structure constant, will be taken up elsewhere. We will not take into account the RG flow of the matter in this paper since in the perturbative regime the renormalization effects coming from the matter sector are small compared to those of pure gravity [51].

This chapter is based on the work [44]. The organization of this chapter is as follows. In Sec. 4.2, we write the RG improved Einstein equation for FLRW cosmology. In Sec. 4.3 we explain our procedure for a particular choice of the relation between the momentum cut-off scale and cosmological time and present our main results for the scale factor and the entropy production rate. In Sec. 4.4, we consider some other choices for the momentum cut-off scale. A summary of the results is given in the last section.

4.2 Improved Field Equations and the cutoff identification

The idea of using the modified conservation law to study cosmology in the context of scale-dependent (or time-dependent) Newton's constant is fairly general and should be widely applicable, even outside the context of asymptotically safe gravity, and for any kind of cosmological metric. In this chapter, we will restrict ourselves and apply the modified conservation law to only the spatially flat FLRW metric,

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] . \tag{4.3}$$

We will take the cosmic matter to be a perfect fluid, for which the energy-momentum tensor is given in eq.(3.5). The improved Einstein equations coming from asymptotically safe gravity can be written as $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G(t)T_{\mu\nu} - \Lambda(t)g_{\mu\nu}$. In writing down these equations, a cut-off identification $k = k(t)$ has been made characterized by the cosmic time t. From these, we obtain a modified Friedmann equation for spatially flat ($K = 0$) FLRW with scale factor $a(t)$ and a modified continuity equation,

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}G(t)\rho + \frac{\Lambda(t)}{3} \tag{4.4}$$

$$\dot{\rho} + 3H(p+\rho) = -\frac{8\pi\rho\dot{G} + \dot{\Lambda}}{8\pi G(t)} . \tag{4.5}$$

While this is written for a perfect fluid, we note that $\dot{G}$ and $\dot{\Lambda}$ will appear in the continuity equation for all types of cosmological matter. A similar equation was written in [61] in terms of the scale k for the case where the matter is a scalar field and the cosmological constant enters through the minimum of the scalar potential. In the consistency approach [51], it is assumed that the left and right hand sides of eq.(4.5) vanish separately. The left hand side of Eq. (4.5) vanishes due to covariant conservation of the energy-momentum tensor $\nabla_\mu T^{\mu\nu} = 0$, which holds when Newton's constant and other coupling constants do not depend on time. The vanishing of the left-hand

and right hand sides of eq.(4.5), that is $\nabla_\mu T^{\mu\nu} = 0$, leads to two separate conditions [42, 51]. For instance, equation (4.5) is resolved as $0 = 0$, meaning that both sides vanish independently

$$\begin{aligned} \dot{\rho} + 3H(p+\rho) &= 0 \\ 8\pi\rho\dot{G} + \dot{\Lambda} &= 0 \end{aligned} \tag{4.6}$$

In this work we shall not consider such a scenario [1]. We used this strategy to thoroughly study the Bianchi-I universe in the preceding chapter. It also be noted that the modified continuity equation in its full generality has been derived in [55, 60, 62, 125–128, 131, 132]. The difference of this work with [131] lies in the investigation of different cut-scale identifications. This is actually interesting since it can test the robustness of the RG improvement procedure.

As discussed above, the energy-momentum tensor will not satisfy $\nabla^\mu T_{\mu\nu} = 0$ in general if G, Λ and any coupling constant for the matter are time-dependent. Instead the entire right hand side of eq.(3.14) is to be considered divergence-free,that is $\nabla^\mu \left[8\pi G(t) T_{\mu\nu} - \Lambda(t) g_{\mu\nu}\right] = 0$, as required by the contracted Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$. This leads to the modified conservation equation, eq.(4.5).

Using $p(t) = \Omega\rho(t)$, the modified conservation equation can be written as

$$8\pi\partial_t \left[G(t)\rho + \frac{\Lambda(t)}{8\pi}\right] = -24\pi(1+\Omega)HG(t)\rho\,. \tag{4.7}$$

Substituting $G(t)\rho$ from eq.(4.4) in the above equation, we obtain

$$\dot{H} = -\frac{1}{2}(3+3\Omega)\left[H^2 - \frac{1}{3}\Lambda(t)\right]. \tag{4.8}$$

In this chapter, we wish to solve the above differential equations at late times. To this end, we shall use the long distance perturbative series expansions of $G(k)$ and $\Lambda(k)$

[1] The vanishing of the right hand side of eq.(4.5) has been called the "consistency condition" in [51]. In our paper we call this consistency condition and eq. (4.5) as modified continuity equation.

as shown in eq.(4.1, 4.2), suitably converted to a time-varying form. As discussed in previous chapter, the identification of the infrared cutoff for momentum scale k involves expressing k in terms of all scales that are relevant to the problem under consideration. In the case of the FLRW universe, homogeneity and isotropy of space-time imply that k is a function of the cosmological time only. This in turn implies that the constants G and Λ take the form $G(t) \equiv G(k = k(t))$, $\Lambda(t) \equiv \Lambda(k = k(t))$. The simplest such behaviour at late times is (we will consider some other possibilities later)

$$k = \frac{\xi}{t} \,. \tag{4.9}$$

The basis of this identification is that when the universe has evolved for a time t , quantum fluctuations with frequencies smaller than $1/t$ would not be important in the cosmology of the universe [51].

Inserting this expression for k in the series for $G(k)$ and $\Lambda(k)$ (eqs.(4.1, 4.2)) , the time dependent Newton's gravitational constant and cosmological constant in the perturbative or low energy regime is found to be

$$G(t) = G_0 \left[1 - \frac{\tilde{\omega} G_0}{t^2} + \mathcal{O}\left(\frac{t_{Pl}^4}{t^4}\right) \right] , \tag{4.10}$$

$$\Lambda(t) = \frac{\tilde{\nu} G_0}{t^4} + \mathcal{O}\left(\frac{t_{Pl}^6}{t^6}\right) , \tag{4.11}$$

where $\tilde{\omega} \equiv \omega \xi^2$, $\tilde{\nu} \equiv \nu \xi^4$ and $t_{Pl} = \sqrt{G_0}$ is the Planck time in natural units. We see that Newton's constant decreases to its classical value G_0 at late times, while the cosmological constant approaches zero, which may be appropriate for our universe. We will use these expressions in the next section to investigate the cosmological expansion and entropy generation in our universe.

4.3 Cosmological expansion and entropy generation

We shall now solve the differential equation for the Hubble parameter (eq.(4.8)). For this, we set $u(t) = \frac{1}{H(t)}$ and rewrite eq.(4.8) in terms of $u(t)$ as

$$\dot{u} = A\left[1 - \frac{\Lambda(t)u^2(t)}{3}\right] \tag{4.12}$$

where $A = \frac{(3+3\Omega)}{2}$. Integrating this equation from $t = t_0$ (present time) to some time t, we obtain

$$u(t) = u_0 + A(t - t_0) - \frac{A}{3}\int_{t_0}^{t} \Lambda(t')u^2(t')\mathrm{d}t' \tag{4.13}$$

where $u_0 \equiv u(t_0)$. This is a Volterra integral equation of the second kind. We now solve this integral equation iteratively since $\Lambda(t)$ is small at large times (eq.(4.11)). For this, we first approximate our solution by taking

$$u(t) \approx u^{(0)}(t) = u_0 + A(t - t_0) \equiv At + B \tag{4.14}$$

where $B = u_0 - At_0$. Substituting this in the integral on the right hand side of eq.(4.13), we obtain to the next approximation

$$u^{(1)}(t) \simeq u^{(0)}(t) - \frac{A}{3}\int_{t_0}^{t} \Lambda(t')\left(u^{(0)}(t')\right)^2 \mathrm{d}t' \,. \tag{4.15}$$

Repeating this process by putting $u^{(1)}(t)$ in the integral in eq.(4.13), we obtain to the next approximation

$$\begin{aligned} u^{(2)}(t) &\simeq u^{(0)}(t) - \frac{A}{3}\int_{t_0}^{t} \Lambda(t')\left(u^{(1)}(t')\right)^2 \mathrm{d}t' \\ &= u^{(0)}(t) - \frac{A}{3}\int_{t_0}^{t} \Lambda(t')\left(u^{(0)}(t')\right)^2 \mathrm{d}t' + 2\left(\frac{A}{3}\right)^2 \int_{t_0}^{t} \Lambda(t_1)\left[\int_{t_0}^{t_1} \Lambda(t_2)\left(u^{(0)}(t_2)\right)^2 \mathrm{d}t_2\right] \mathrm{d}t_1 \\ &\qquad + \mathcal{O}(\Lambda^3(t)) \,. \end{aligned} \tag{4.16}$$

Substituting $\Lambda(t)$ from eq.(4.11) in eq.(4.16) and keeping terms up to $\mathcal{O}(G_0)$, we get

$$u^{(1)}(t) \simeq u^{(0)}(t) - \frac{A\tilde{\nu}G_0}{3}\left[-\frac{A^2}{t} - \frac{AB}{t^2} - \frac{B^2}{3t^3} + \frac{A^2}{t_0} + \frac{AB}{t_0^2} + \frac{B^2}{3t_0^3}\right]. \quad (4.17)$$

Defining $C = \left[\frac{A^2}{t_0} + \frac{AB}{t_0^2} + \frac{B^2}{3t_0^3}\right]$ for convenience, we get

$$u(t) \simeq At + B + \frac{A\tilde{\nu}G_0}{3}\left[\frac{A^2}{t} + \frac{AB}{t^2} + \frac{B^2}{3t^3} - C\right]. \quad (4.18)$$

This now gives a differential equation for the scale factor $a(t)$. This reads

$$\begin{aligned} H = \frac{\dot{a}}{a} &= \frac{1}{At+B}\left[1 + \frac{A\tilde{\nu}G_0}{3(At+B)}\left(\frac{A^2}{t} + \frac{AB}{t^2} + \frac{B^2}{3t^3} - C\right)\right]^{-1} \\ &= \frac{1}{At+B}\left[1 - \frac{\tilde{\nu}G_0}{3(t+\frac{B}{A})}\left(\frac{A^2}{t} + \frac{AB}{t^2} + \frac{B^2}{3t^3} - C\right) + ...\right]. \end{aligned} \quad (4.19)$$

Integrating the above equation, we get upto $\mathcal{O}(\frac{t_{Pl}^2}{t^2})$

$$\ln a(t) = \frac{1}{A}\ln\left(t + \frac{B}{A}\right) - \frac{A\tilde{\nu}G_0}{3}\left[\frac{A}{B\left(t+\frac{B}{A}\right)} + \frac{A^2\ln t}{B^2} - \frac{A^2\ln(t+\frac{B}{A})}{B^2}\right] - \frac{C\tilde{\nu}G_0}{3A\left(t+\frac{B}{A}\right)} + \ln\tilde{a} \quad (4.20)$$

where $\ln\tilde{a}$ is a constant of integration.

Now we use the initial condition $u(0) = 0$. Then it follows that $B = 0$ and

$$t_0 = \frac{1}{AH_0} = \frac{2}{(3+3\Omega)H_0} \quad (4.21)$$

where t_0 is the age of the universe and $H_0 = \frac{1}{u_0}$ is the present Hubble parameter. Now noting that the second term in eq.(4.20) takes the value $\frac{A\tilde{\nu}G_0}{6t^2}$ when $B \to 0$, we can write the final form of the scale factor $a(t)$ as

$$\begin{aligned} a(t) &= \tilde{a}t^{1/A}\exp\left(\frac{A\tilde{\nu}G_0}{6t^2} - \frac{C\tilde{\nu}G_0}{3At}\right) \\ &= \tilde{a}t^{1/A}\left(1 - \frac{A\tilde{\nu}G_0}{3t_0 t} + \frac{A\tilde{\nu}G_0}{6t^2} + \mathcal{O}(G_0^2)\right) \end{aligned} \tag{4.22}$$

where in the last line we have used $C = \frac{A^2}{t_0}$ since $B = 0$.

The constant of integration $\tilde{a}$ can be fixed by remembering that $a(t_0) = 1$. This yields

$$\tilde{a} = \frac{1}{t_0^{1/A}}\left(1 - \frac{A\tilde{\nu}G_0}{6t_0^2} + \mathcal{O}(G_0^2)\right)^{-1} . \tag{4.23}$$

Eq.(4.22) gives the quantum correction in the scale factor.

Substituting $\Lambda(t)$ from eq.(4.11) and the Hubble parameter from eq.(4.19) (for $B = 0$) in eq.(4.4), and putting $A = \frac{(3+3\Omega)}{2}$, we get the quantum corrected energy density to be

$$\rho = \frac{3}{8\pi}\left(\frac{2}{3+3\Omega}\right)^2 \frac{1}{G_0 t^2}\left[1 + \frac{(3+3\Omega)^2\tilde{\nu}G_0}{6t_0 t} + \frac{\tilde{\omega}G_0}{t^2} - \frac{(3+3\Omega)^2\tilde{\nu}G_0}{4t^2} + \mathcal{O}(G_0^2)\right] . \tag{4.24}$$

Note that the first term is exactly the classical result of energy density in FLRW cosmology. The subsequent terms are the quantum corrections in the energy density up to $\mathcal{O}(\frac{t_{Pl}^2}{t^2})$.

Now we proceed to discuss entropy generation in the above scenario in line with the arguments in [131, 132]. From the modified continuity equation (4.5), we get

$$[\dot{\rho} + 3H(p+\rho)]\,\frac{4\pi}{3}R_0^3 a^3 = \frac{4\pi}{3}R_0^3\tilde{\mathcal{P}}(t)\,, \tag{4.25}$$

where we have written

$$\tilde{\mathcal{P}}(t) = -\left[\frac{\dot{\Lambda} + 8\pi\rho\dot{G}}{8\pi G}\right]a^3\,, \tag{4.26}$$

with R_0 being the radius of the universe at the present time t_0. Note that the scale factor here is dimensionless with $a(t) = \frac{R(t)}{R_0}$. We can rewrite the above equation as

$$\frac{\mathrm{d}(\frac{4\pi}{3}\rho R_0^3 a^3)}{\mathrm{d}t} + p\frac{\mathrm{d}(\frac{4\pi}{3}R_0^3 a^3)}{\mathrm{d}t} = \frac{4\pi}{3}R_0^3\tilde{\mathcal{P}}(t)\,. \tag{4.27}$$

In terms of energy $U = \frac{4\pi}{3}\rho R_0^3 a^3$ and proper volume $V = \frac{4\pi}{3}R_0^3 a^3$, the above equation takes the form

$$\frac{\mathrm{d}U}{\mathrm{d}t} + p\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{4\pi}{3}R_0^3\tilde{\mathcal{P}}(t)\,. \tag{4.28}$$

Using the standard thermodynamic relation $\mathrm{d}U + p\mathrm{d}V = T\mathrm{d}S$, the above equation can be recast as

$$T\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{4\pi}{3}R_0^3\tilde{\mathcal{P}}(t)\,, \tag{4.29}$$

where S is the entropy carried by the perfect fluid inside the comoving volume $V = \frac{4\pi}{3}R_0^3 a^3$. In classical cosmology, $\tilde{\mathcal{P}}(t) = 0$ as Λ and G do not depend on time. Therefore we can conclude that the entropy does not change during the expansion of the universe. Also note that $\tilde{\mathcal{P}}(t) = 0$ in the consistency equation approach [51] as $\dot{\Lambda} + 8\pi\rho\dot{G} = 0$, so that the entropy production rate vanishes.

From eqs.(4.28, 4.29), the change in entropy reads

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{4\pi}{3}R_0^3\frac{\tilde{\mathcal{P}}(t)}{T} = \frac{4\pi}{3}R_0^3\left[\dot{\rho} + 3H(\rho + p)\right]\frac{a^3}{T}\,. \tag{4.30}$$

To calculate the entropy or the change in entropy, we need to know the temperature T as a function of time which is in principle unknown. We also require the relation between p and ρ which we have taken to be $p = \Omega\rho$ in our analysis. Assuming radiation dominance, the energy density is taken to be $\rho(t) = \sigma^4 T^4$, where $\sigma \equiv \left(\frac{\pi^2 n_{\text{eff}}}{30}\right)^{\frac{1}{4}}$, with n_{eff} being given by $n_{\text{eff}} = n_b + \frac{7}{8}n_f$, where n_b and n_f are the bosonic and fermionic massless degrees of freedom respectively. This is in line with the argument in [131] which mentions that significant entropy production takes place only in the radiation dominated universe. Furthermore, since the non-adiabaticity is small, the Stefan-Boltzmann relation among p, ρ and T are still valid in the non-equilibrium cosmological scenario with entropy production [133, 134].

Using the equation of state $p = \frac{\rho}{3}$, the entropy production rate in terms of the scale factor and energy density is given by

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{16\pi}{9}\sigma R_0^3 a^3 \rho^{3/4}\right] . \tag{4.31}$$

Integrating eq.(4.31), the final entropy carried by proper volume V reads

$$S(t) = \frac{16\pi}{9}\sigma R_0^3 a^3 \rho^{3/4} + S_c\,, \tag{4.32}$$

where S_c is a constant of integration. Note that the radiation energy density obeys $a^3\rho^{3/4} = \textit{constant}$ if G and Λ are independent of time or obeys the consistency condition and hence the entropy S is a constant. Otherwise, the quantity $a^3\rho^{3/4}$ is a function of time and hence the entropy S is time dependent.

Substituting the expression for the scale factor $a(t)$ and the energy density $\rho(t)$ from eqs.(4.22, 4.24) in the above equation, we obtain the entropy for $\Omega = \frac{1}{3}$ to be

$$S(t) = \frac{16\pi}{9}\sigma R_0^3 \tilde{a}^3 \left[\frac{3}{32\pi G_0}\right]^{\frac{3}{4}} \left[1 + (3\tilde{\omega} - 8\tilde{\nu})\frac{G_0}{4t^2} + \cdots\right] + S_c\,. \tag{4.33}$$

From the above result, we observe that the leading quantum correction to the entropy for the radiation dominated universe vanishes if $(3\tilde{\omega} - 8\tilde{\nu}) = 0$, i.e. for $\xi^2 = \frac{3\omega}{8\nu}$. This is the same value of ξ that one gets from the consistency approach [51] in which the energy momentum tensor is taken to be covariantly conserved, independent of the scale(time)-dependence of G and Λ. In such a scenario the entropy production rate is zero and hence the expansion of the universe is adiabatic. We can also obtain an upper bound for the parameter ξ from the positivity of the entropy at all times which imposes $(3\tilde{\omega} - 8\tilde{\nu}) \geq 0$ leading to $\xi \leq \sqrt{\frac{3\omega}{8\nu}}$.

4.4 More choices of cut-off

In the analysis of the previous section, we chose the energy scale to be inversely proportional to time. However, there can be other choices. An interesting possibility that was considered in [131] is to take the energy scale as a function of the Hubble parameter instead of cosmological time. In this section, we obtain solutions for the scale factor for some power law relations between the energy scale and the Hubble parameter. These choices are natural since in an FLRW geometry, the Hubble parameter measures the curvature of space-time. The cut-off choices mean that the quantum fluctuations with momentum smaller than the curvature of space-time at that particular Hubble parameter are not integrated out in constructing the effective field theory. Of course, there may be other possible choices of cut-off. There is another reason behind these choices since they are amenable to analytical treatment. Note that the choice of these cut-offs introduces a dimensionful constant. As was mentioned in [51], subleading quantum effects may necessitate cut-offs which are functions of t, $a(t)$, $\dot{a}(t)$, etc. Clearly, in such cases additional dimensionful constants will appear.

4.4.1 Cut-off $k = \varepsilon H^{\frac{3}{4}}$

Of course, for a general cut-off function it is not possible to find closed-form solutions for the scale factor. Let us consider a special case for which a solution can be found, $k = \varepsilon H^{\frac{3}{4}}$, with ε being a dimensionful constant. This choice in turn means that the cut-off k is a function of $a(t)$ and $\dot{a}(t)$. These choices may capture more subtle subleading effects which would be missed by the cut-off $k = \xi/t$ [51]. We use this cut-off to find an expression for Λ from eq.(4.2),

$$\Lambda(t) = \bar{\nu} G_0 H^3 + \cdots \tag{4.34}$$

where $\bar{\nu} = \nu\varepsilon^4$. Putting $\Lambda(t)$ from eq.(4.34) up to order $\mathcal{O}(G_0)$ in eq.(4.8), we get the differential equation for the Hubble parameter to be

$$\dot{H} = -\frac{(3+3\Omega)}{2}\left[H^2 - \frac{\bar{\nu}G_0H^3}{3}\right] . \tag{4.35}$$

We first express H as a function of the scale factor $a(t)$. For this we note that $\frac{1}{H}\left(\frac{\mathrm{d}H}{\mathrm{d}t}\right) = \left(\frac{\mathrm{d}H}{\mathrm{d}a}\right)\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)\frac{1}{H} = a\left(\frac{\mathrm{d}H}{\mathrm{d}a}\right)$ and recast the above equation in the form

$$a\left(\frac{\mathrm{d}H}{\mathrm{d}a}\right) = -\frac{(3+3\Omega)}{2}H\left[1 - \frac{\bar{\nu}G_0H}{3}\right] . \tag{4.36}$$

Integrating the above equation from the present time t_0 to some time t, we obtain

$$\frac{H\left(1 - \frac{\bar{\nu}G_0}{3}H_0\right)}{H_0\left(1 - \frac{\bar{\nu}G_0}{3}H\right)} = a^{-\frac{(3+3\Omega)}{2}} \tag{4.37}$$

where we have set $a(t_0) = 1$ and written $H(t_0) = H_0$ for the value of the Hubble parameter at the present time. Solving for H, we obtain

$$H = H_0a^{-\frac{(3+3\Omega)}{2}}\left[1 + \frac{\bar{\nu}G_0H_0}{3}\left(a^{-\frac{(3+3\Omega)}{2}} - 1\right)\right]^{-1} . \tag{4.38}$$

Solving this equation gives the scale factor $a(t)$ as

$$\left(1 - \frac{\bar{\nu}G_0H_0}{3}\right)\left(a^{\frac{3+3\Omega}{2}} - 1\right) + \frac{(3+3\Omega)}{2}\frac{\bar{\nu}G_0H_0}{3}\ln a = \frac{(3+3\Omega)}{2}H_0\left(t - t_0\right) , \tag{4.39}$$

where the constant of integration has been fixed from the condition $a(t_0) = 1$ and we have assumed that $\Omega \neq -1$. The functional dependence of the scale factor a in terms of the cosmological time t can be obtained iteratively, and up to first order in G_0 reads

$$\begin{aligned} a(t) \;=\; & \left[1 + \frac{(3+3\Omega)}{2}H_0(t-t_0) + \frac{\bar{\nu}G_0H_0}{3}\right. \\ & \left. \times\left(\frac{(3+3\Omega)}{2}H_0(t-t_0) - \ln\left(\frac{(3+3\Omega)}{2}H_0(t-t_0) + 1\right)\right)\right]^{\frac{2}{(3+3\Omega)}} . \end{aligned} \tag{4.40}$$

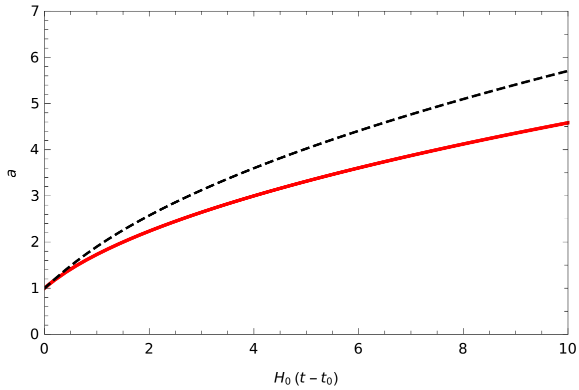


Fig. 4.1 Scale factor versus time for a radiation dominated universe. Solid line : classical scale factor. Dashed line : quantum corrected scale factor.

Note that at late times, i.e. as $t \to \infty$, we find the usual behaviour of classical cosmology, $a \to \infty$ and $H \to 0$.

To get an idea about the energy scale of the cut-off k, we estimate the dimensionful constant ε appearing in the cut-off using eq.(4.34). Using the observed values $H_0 \simeq 1.45 \times 10^{-42}$ GeV, $G_0 \simeq 0.67 \times 10^{-38}$ GeV^{-2} [135] and $\Lambda_0 \simeq 4.30 \times 10^{-84}$ GeV^2 [136] at the present time, we find $\varepsilon^4 \simeq 11.01 \times 10^{80}$ GeV. It is interesting to note that this energy scale is of the same order of magnitude as the total mass of the universe. For these values, $\frac{\bar{\nu} G_0 H_0}{3}$ turns out to be a number of order unity. Restoring $\hbar$ and c, we find that this number is $\frac{\hbar \bar{\nu} G_0 H_0}{3c}$ which is clearly a quantum correction.

We use this to make a plot of the scale factor $a(t)$ *vs.* $H_0(t-t_0)$ for radiation dominated universe ($\Omega = \frac{1}{3}$) which is displayed in Fig.(4.1). We observe that although the quantum corrections are smaller than the classical solutions, they become larger with increasing cosmological time. This is counter-intuitive since quantum gravity effects are expected to be important at early times and not at late times. The exhibited behaviour may owe its origin to the choice of the cut-off since for this choice, the cosmological time coupled to the quantum correction appears in the numerator in the right hand side of eq.(4.40). This is in contrast to the $k = \xi/t$ cut-off choice where the cosmological time coupled to the quantum correction appears in the denominator. For the calculation of entropy, we note that using the cut-off $k = \varepsilon H^{\frac{3}{4}}$ in eq.(4.1), we get

$$G(t) = G_0 \left[1 - \bar{\omega} G_0 H^{\frac{3}{2}} + \cdots\right] \tag{4.41}$$

where $\bar{\omega} = \omega\varepsilon^2$. It is clear that $G \to G_0$ at late times. Using this form of $G(t)$, we get the energy density in terms of the Hubble parameter to be

$$\rho(a) = \frac{3H^2}{8\pi G_0}\left[1 + \bar{\omega} G_0 H^{\frac{3}{2}} + \cdots\right]\left[1 - \frac{\bar{\nu} G_0}{3} H\right]. \tag{4.42}$$

Using the expression for H in eq.(4.42), we obtain the energy density in terms of the scale factor $a(t)$ as

$$\rho(a) = \frac{3H_0^2}{8\pi G_0} a^{-(3+3\Omega)}\left[1 + \bar{\omega} G_0 H_0^{\frac{3}{2}} a^{-\frac{3(3+3\Omega)}{4}} - \frac{\bar{\nu} G_0 H_0}{3}\left(2a^{-\frac{3+3\Omega}{2}} - 1\right) + \cdots\right]. \tag{4.43}$$

The entropy produced can now be calculated in case of the radiation dominated universe by setting $\Omega = \frac{1}{3}$. We find

$$S(a) = \frac{16\pi}{9}\sigma R_0^3 \left(\frac{3H_0^2}{8\pi G_0}\right)^{\frac{3}{4}}\left[1 + \frac{3\bar{\omega} G_0 H_0^{\frac{3}{2}}}{4a^3} - \frac{\bar{\nu} G_0 H_0}{4}\left(\frac{2}{a^2} - 1\right) + \cdots\right] + S_c\,. \tag{4.44}$$

The entropy approaches a constant value at late times.

4.4.2 Cut-off $k = \gamma H$

A cut-off identification suggested in [131] was $k(t) = \gamma H$, based on the fact that the Hubble parameter measures the curvature of spacetime in a FLRW geometry. For the sake of completeness, we shall now calculate the Hubble parameter and the entropy production for this case. We now have $\Lambda(t) = \breve{\nu} G_0 H^4 + \cdots$ and $G(t) = G_0\left[1 - \breve{\omega} G_0 H^2 + \cdots\right]$ where $\breve{\nu} = \nu\gamma^4$ and $\breve{\omega} = \omega\gamma^2$. The differential equation (4.8) for $H(t)$ for this cut-off reads

$$a\left(\frac{\mathrm{d}H}{\mathrm{d}a}\right) = -\frac{(3+3\Omega)}{2} H\left[1 - \frac{\breve{\nu} G_0 H^2}{3}\right]. \tag{4.45}$$

This has the solution

$$H = H_0 a^{-(3+3\Omega)/2}\left[1 - \frac{\breve{\nu} G_0 H_0^2}{3}(1 - a^{-(3+3\Omega)})\right]^{-\frac{1}{2}} . \qquad (4.46)$$

Solving this equation and keeping terms up to first order in G_0 gives the scale factor $a(t)$ as (for $\Omega \neq -1$)

$$\left(a^{\frac{3+3\Omega}{2}} - 1\right) - \frac{\breve{\nu} G_0 H_0^2}{6}\left(a^{\frac{3+3\Omega}{2}} + a^{-\frac{3+3\Omega}{2}} - 2\right) = \frac{(3+3\Omega)}{2} H_0 (t - t_0) . \qquad (4.47)$$

We perturbatively calculate the functional dependence of the scale factor a in terms of the cosmological time t, and up to first order in G_0 we find

$$\begin{aligned} a(t) &= \left[\frac{(3+3\Omega)}{2} H_0(t - t_0) + 1 + \frac{\breve{\nu} G_0 H_0^2}{6} \right. \\ &\quad \left. \times \left(\frac{(3+3\Omega)}{2} H_0(t-t_0) + \frac{1}{\frac{(3+3\Omega)}{2} H_0(t-t_0) + 1} - 1\right)\right]^{\frac{2}{(3+3\Omega)}} . \end{aligned} \qquad (4.48)$$

Once again we find that $a \to \infty$ as $t \to \infty$, and therefore $H \to 0$. We can estimate the dimensionless constant γ appearing in the cut-off using eq.(4.2). As before, we use the observed values of the constants Λ_0, H_0 and G_0 at the present time, and thus calculate $\gamma^4 \simeq 7.59 \times 10^{122}$. This in turn makes $\frac{\breve{\nu} G_0 H_0^2}{6}$ a number of order unity. Using these numbers, we have plotted the scale factor $a(t)$ against $H_0(t - t_0)$ for the radiation dominated universe, which is displayed in Fig.(4.2).

Using the expression for H obtained in eq.(4.46) we calculate the energy density as a function of the scale factor

$$\rho(a) = \frac{3H_0^2}{8\pi G_0} a^{-(3+3\Omega)}\left[1 + \frac{2\breve{\nu} G_0 H_0^2}{3}\left(\frac{1}{2} - a^{-(3+3\Omega)}\right) + \breve{\omega} G_0 H_0^2 a^{-(3+3\Omega)} + \cdots\right] . \qquad (4.49)$$

From this result, we get the entropy in the radiation dominated universe as

$$S(a) = \frac{16\pi}{9}\sigma R_0^3 \left(\frac{3H_0^2}{8\pi G_0}\right)^{\frac{3}{4}}\left[1 + \frac{3\breve{\omega} G_0 H_0^2}{4a^4} + \frac{\breve{\nu} G_0 H_0^2}{2}\left(\frac{1}{2} - a^{-4}\right) + \cdots\right] + S_c . \qquad (4.50)$$

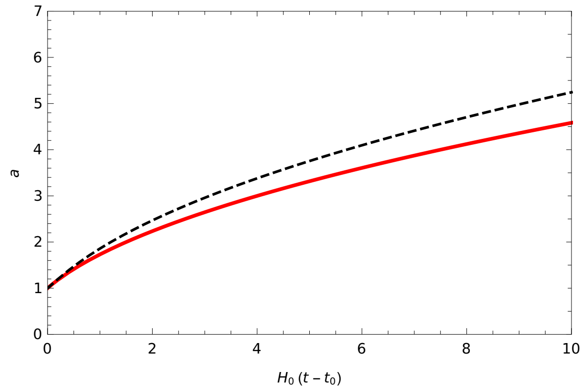


Fig. 4.2 Scale factor versus time for a radiation dominated universe. Solid line : classical scale factor. Dashed line : quantum corrected scale factor.

The entropy is again a constant at late times.

4.5 Summary and Conclusions

The asymptotic safety approach to quantum gravity results in the running of Newton's gravitational constant G and the cosmological constant Λ. In this chapter, we have studied the consequence of this scale dependence on the FLRW universe, specifically, the late time behaviour of the scale factor using different choices of the infrared cut-off scale k. Our approach is different from earlier ones [42, 51] (but in line with earlier work [60, 131, 132]) in that we have considered the conservation of the entire right hand side of Einstein equation.

We consider different interpretations of the cut-off scale k in terms of the cosmological time parameter, as is typically done in relating the scale dependent results to cosmological solutions. The first choice of the cut-off scale we take is $k = \frac{\xi}{t}$. Incorporating this into the late time relations for the running of G and Λ, we solve the differential equations for the Hubble parameter and the energy density using an iterative approach. The solutions give the quantum corrections for the scale factor and the energy density. We observe that an $\mathcal{O}(1/t)$ term appears in the quantum correction of the scale factor. This term did not appear in [51] where the covariant conservation of the energy momentum tensor was imposed separately. Clearly, this behaviour is captured only if the scale (time) dependence of the gravitational and cosmological "constants" is taken into account for the conservation of the energy-momentum tensor.

We have also calculated the entropy production rate at late times for different choices of cut-off, using methods similar to [131, 132]. For this we have assumed a radiation dominated universe and also that the non-adiabaticity is small. Here, we observe that the entropy gets a time dependent quantum correction at $\mathcal{O}(1/t^2)$. We therefore conclude that if $T_{\mu\nu}$ is taken to be conserved independently of the scale (time)-dependence of G and Λ, the entropy production rate would vanish as a result.

Considering the possibility that the cut-off scale may have a simpler functional relationship with the Hubble parameter than with the cosmological time, we looked at two other choices of the cut-off scale, namely, $k = \varepsilon H^{\frac{3}{4}}$, and $k = \gamma H$. The time evolution of the scale factors with these cut-off choices is one of the main findings in this paper. Further, this study also gives an idea about the dependence of evolution of the universe on the cut-off choices. In particular, we find that the choices $k = \varepsilon H^{\frac{3}{4}}$, $k = \gamma H$ lead to qualitatively similar physical results thereby validating the robustness of the RG procedure. We have then calculated the quantum corrected energy density and entropy in terms of the scale factor for these choices of the cut-off scale. We would also like to comment that for the cut-offs $k = \varepsilon H^{\frac{3}{4}}$ and $k = \gamma H$ the behaviours are similar to the classical case, and entropy becomes constant at late times.

The approach of the modified continuity equation will be useful in situations where the RG flow of the matter sector is included with that of gravity. For example, for electromagnetic radiation one should also include the scale dependence of the fine structure constant which in this picture translates to a time dependence. Time-varying G and Λ have been considered even outside asymptotically safe gravity [114, 117, 137], while cosmologies in which the fine structure constant varies with time have been considered in [138–140]. We expect this approach will be useful in dealing with the time variations of the "constants" in such models.

Chapter 5

Black hole thermodynamics in asymptotically safe gravity

5.1 Introduction

As mentioned earlier, black hole thermodynamics plays an important role in providing fundamental fundamental hints about the quantum aspect of gravity. Hawking and Bekenstein showed that black hole radiates with a temperature. They also found that these objects have entropy given by $S = \frac{A_{class}}{4\hbar G_0}$ where A_{class} is the area of the event horizon[4–6]. These findings indicate that black hole thermodynamics should be investigated in the context of quantum gravity since both Planck length and Newton's gravitational constant appear in the expression of entropy. In [141], analogies to the zeroth and first law of black hole thermodynamics were found where temperature was associated with the strength of the gravitational field at event horizon called surface gravity. It was Hawking who had taken quantum effects of matter fields in curved spacetime to show that black holes radiate at a temperature and have an entropy [7].

A semi-classical approach is usually taken where the quantum effects are included through the matter fields and gravity is treated in a classical way. This approach breaks down at a Planck scale where quantum gravity effects become very important. To capture quantum gravity effects in thermodynamics, there are phenomenological theories, namely, the generalized uncertainty relation (GUP) [142–146], modified dispersion relation with rainbow gravity [147–150], tunnelling method [151, 152]. Asymptotic safe (AS) gravity is a promising alternative to a consistent quantum theory of the gravitational field. We have discussed in earlier chapters that AS requires the existence of a nontrivial fixed point which controls the behaviour of the coupling constants. In this approach, physical quantities are safe from divergences in the ultraviolet regime, without being perturbatively renormalizable.

As discussed, the primary tool for investigating the theory is functional renormalization group equation (FRGE) [26, 31, 95, 99]. This idea corresponds to successively integrating out the momentum in a path integral formalism. This method permits a continuous interpolation through the renormalization group trajectories between the microscopic and macroscopic degree of freedom controlled by the effective action Γ_k. These trajectories satisfy a functional renormalization group equation. This results in the running of the Newton's gravitational constant.

By setting the "running" Newton's constant in place of Newton's constant, the renormalization group improved Schwarzschild metric has been proposed identifying $G(k) \equiv G(r)$ [111]. Here the cut off scale k is identified with a position dependent quantity, namely, the proper distance $d(r)$. The parameter γ helps to identify the cut-off scale properly. The parameter γ has the value $\gamma = \frac{9}{2}$ if $k = \frac{\zeta}{d(r)}$ and $\gamma = 0$ corresponds to $k = \frac{\xi}{r}$ [153].

In this chapter, we will investigate the thermodynamics and phase transition for RG improved Schwarzschild metric. First, we explore the modification of the thermodynamic quantities, namely, Hawking temperature, heat capacity and entropy due to the presence of the general parameter γ. We will compare the differences in the thermodynamic quantities qualitatively and quantitatively between $\gamma = 0$ and $\gamma = \frac{9}{2}$. Then, we shift attention to investigate phase transition and thermodynamic stability considering the black hole inside a finite spherical concentric cavity whose radius is larger than the horizon radius of the black hole. Here we calculate the on-shell free energy and observe that the black hole state always prevails for all temperatures incorporating quantum corrections. We have surprisingly found through our analysis that there is no Hawking-Page phase transition in the quantum corrected Schwarzschild black hole kept in a concentric spherical cavity, which is in complete contrast to the usual Schwarzschild black hole scenario kept in a cavity where one gets Hawking-Page phase transition.

This chapter is based on the work [45]. It is organized in the following way. In Sec. 5.2, we briefly discuss the basic of thermodynamics and four laws of black hole. In Sec. 5.3, we describe the set up for quantum improved Schwarzschild black hole. In Sec. 5.4, we study the modification of the thermodynamic properties for RG improved Schwarzschild metric. In Sec. 5.5, we investigate how mass of the black evaporate with time. In Sec. 5.6, we investigate the phase transition and thermodynamic stability of the black hole. We conclude thereafter.

5.2 Black hole Thermodynamics

In the classical theory of general relativity, the area of the event horizon never shrinks, which was the first revelation that black holes have thermodynamic characteristics. It is a clear-cut analogy with the second law of thermodynamics where the area of the event horizon acting as the entropy [154]. However, a black hole abides by all thermodynamic-like laws in addition to the second law. A thermodynamic system can be characterized without knowing the position and the momentum of every molecule, according to thermodynamics. A few macroscopic variables, like temperature and pressure, are all that are needed to describe the system. According to No Hair Theorem [155, 156], only three classical, externally observable parameters, namely, mass M, electric charge Q, and angular momentum J can fully define all stationary asymptotically flat black hole solutions in general relativity. This implies that the black hole is a coarse-grained description in some way. So that, the black hole laws can be compared with the thermodynamic laws. In the zeroth and first laws of black hole thermodynamics, the role of temperature was represented by a quantity called the surface gravity κ which evaluated the gravitational force (per unit mass) or the gravitational acceleration on the horizon measured by asymptotic observer [141]. Due to the similarities of the black hole laws with the thermodynamics, Bekenstein proposed that the physical entropy of the black hole should be specified as a multiple of the event horizon's area, measured in Planck units [5]. This idea would result in contradictions and violations of the second law, if, as was assumed at the time, black holes could swallow particles but could not release anything. Then, at any non-zero temperature, black holes could not exist in equilibrium with thermal radiation. But when it was revealed that a black hole would produce and emit particles like a hot body with a temperature of $\kappa/2\pi$ taking the quantum effects into account, the above mentioned obstacle was eliminated [7]. Now from the first law, it was evident that a black hole's entropy was $S = \frac{A}{4l_{Pl}^2}$, where A is denoted the area of the event horizon and $l_{Pl} = \sqrt{G_0}$ is the Planck's length with $\hbar = c = k_B = 1$. After realising that these thermodynamic characteristics were a result of the periodicity in the imaginary time coordinate required to eliminate the conical singularity in the Euclidean version of of black hole metrics, it became possible to comprehend them more thoroughly. The mathematical formulation of computing the Hawking temperature is described below starting with the requirement that the Euclidean space time must be smooth.

Hawking temperature and Euclidean formalism : In the Euclidean formalism for obtaining the Hawking temperature, one requires that the temperature is the inverse of the period which can be calculated with a periodic identification of imaginary time [157–159]. For this considering the generic black hole metric which is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d{\Omega_2}^2 \tag{5.1}$$

where $f(r_+) = g(r_+) = 0$ at the horizon and Ω_2 is the unit volume of 2 dimensional sphere. By identifying time with Euclidean signature $t = i\tau$, the period of τ leads to the temperature T as

$$\tau \rightarrow \tau + \frac{1}{T} \, . \tag{5.2}$$

In Euclidean signature, the metric (5.1) is given by

$$ds_E^2 = f(r)d\tau^2 + \frac{dr^2}{g(r)} + r^2 d\Omega_2{}^2 \ . \tag{5.3}$$

Now one can approximate $f(r) \approx f'(r_+)\,(r - r_+)$ and $g(r) \approx g'(r_+)\,(r - r_+)$ near the horizon. So that, the Euclidean metric reads

$$\begin{aligned} ds_E^2 &= f'(r_+)\,(r - r_+)\,d\tau^2 + \frac{dr^2}{g'(r_+)\,(r - r_+)} + r_+^2 d\Omega_2{}^2 & (5.4) \\ &= \varrho^2 d\left(\frac{f'(r_+)g'(r_+)}{2}\tau\right)^2 + d\varrho^2 + r_+^2 d\Omega_2{}^2 \\ &= \varrho^2 d\theta^2 + d\varrho^2 + r_+^2 d\Omega_2{}^2 & (5.5) \end{aligned}$$

where a new coordinate is introduced in the form $\varrho \equiv \frac{2}{\sqrt{g'(r_+)}}\sqrt{(r - r_+)}$. The metric in this coordinate resembles a plane in polar coordinates where $\theta \equiv \frac{f'(r_+)g'(r_+)}{2}\tau$ should have period 2π to avoid conical singularity at $\varrho = 0$. Then the period of τ reads

$$\begin{aligned} &\theta \to \theta + 2\pi \\ \Rightarrow \ &\tau \to \tau + \frac{4\pi}{\sqrt{f'(r_+)g'(r_+)}} \ . \end{aligned} \tag{5.6}$$

Comparing eq.(5.2) and eq.(5.6), the Hawking temperature is obtained as

$$\begin{aligned} T &= \frac{1}{4\pi}\sqrt{f'(r_+)g'(r_+)} \\ &= \frac{\kappa}{2\pi} \end{aligned} \tag{5.7}$$

where κ is the surface gravity for static, spherically symmetric black holes. As mentioned before, the surface gravity is the gravitational acceleration on the horizon measured by an asymptotic observer. The local acceleration is defined by

$$a = \frac{1}{\varrho} \ . \tag{5.8}$$

So the surface gravity or the acceleration measured on the horizon by an infinitely distance observer is connected by

$$\kappa \equiv a_\infty = \sqrt{\frac{g_{00}(r)}{g_{00}(\infty)}}\, a = \frac{\sqrt{f'(r_+)g'(r_+)}}{2}\ . \tag{5.9}$$

This mathematical identification between surface gravity and black hole temperature plays the crucial role in order to understand thermodynamic behaviour of a black hole.

Laws of black hole thermodynamics : The "The four laws of black hole mechanics" was published by Bardeen, Carter, and Hawking [141] for a stationary asymptotically flat black hole in four dimensions, that has a mass M , an angular momentum J, and a charge Q. These laws state :

Zeroth law : It says that the surface gravity, κ is constant over the event horizon. Here a stationary black hole with constant horizon gravity is in a state of equilibrium.

First law : When the parameters M, J, and Q of a stationary black holes differ by small variations, then the change in energy is written as

$$dM = \frac{\kappa}{8\pi G_0} dA + \Omega_H dJ + \phi_H dQ \tag{5.10}$$

where Ω_H and ϕ_H are the angular velocity and electric charge at the horizon respectively. For Schwarzschild black hole ($Q = 0$ and $J = 0$), the mass of the black hole increases by the relation

$$dM = \frac{\kappa}{8\pi G_0} dA = \frac{dA}{32\pi G_0^2 M}\ . \tag{5.11}$$

Second law : As we mentioned earlier, Hawking demonstrated that the black hole can emit radiation at a temperature of $T = \frac{\kappa}{2\pi r}$ taking quantum fields in a black hole background. When this is understood, $\frac{A}{4G_0}$ should be interpreted as the black hole's actual entropy. Then, Bekenstein proposed a generalised second law which states that the entropy of matter and black holes together never decreases:

$$dS_{tot} = d\left(S_{matter} + S_{BH}\right) = d\left(S_{matter} + \frac{A}{4G_0}\right) \geq 0 \ . \tag{5.12}$$

Third law (Nernst's law) : It states that surface gravity κ cannot be reduced to zero in a finite number of steps using any procedures.

This discussion will help us to investigate further the thermodynamic properties of black holes which will be discussed in the following section.

5.3 Quantum improved Schwarzschild black hole

We consider a static spherically symmetric black hole spacetime taking into account the quantum gravitational effects. For this, the key ingredient is the running Newton's constant which is obtained in the formalism of the renormalization group (RG) flow of the effective average action for gravity [26]. The RG improved Schwarzschild metric reads [111]

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2{}^2 \tag{5.13}$$

where

$$f(r) = 1 - \frac{2G(r)M}{r} \ . \tag{5.14}$$

Note that here the leading quantum corrections have been considered by improving solutions of the classical Einstein equation. By replacing G_0 with $G(r)$, the quantum improved Schwarzschild metric is obtained.

We now briefly review the procedure of obtaining the position dependent Newton's constant $G(r)$ from the running $G(k)$ through the identification of the infrared cut-off scale k [111]. The cut-off scale k was chosen to be $k = \frac{\zeta}{d(r)}$ in terms of a position dependent function. The distance scale $d(r)$ is identified with the proper distance from the point P (Schwarzschild coordinate (t, r, θ, ϕ)) to the centre of the black hole along some curve at least for the spherical symmetric case.

The approximate analytical solution for the dimensionful running Newton's constant $G(k) \equiv \frac{\tilde{g}(k)}{k^2}$ is discussed in subsection 2.2.3 of chapter 2 which reads

$$G(k) = \frac{G_0}{1 + \omega G_0 k^2} \ . \tag{5.15}$$

Next to calculate the position dependent form of $G(k)$, one takes the form of the interpolating proper distance $d(r)$ to be [111]

$$d(r) = \left(\frac{r^3}{r + \gamma G_0 M} \right)^{1/2} \ . \tag{5.16}$$

This form of $d(r)$ is chosen since for large r

$$d(r) = r[1 + \mathcal{O}(1/r)] \tag{5.17}$$

and for small r

$$d(r) = \frac{r^{3/2}}{\sqrt{\gamma G_0 M}} + \mathcal{O}(r^{5/2}) \ . \tag{5.18}$$

For two curves, namely, a straight radial line from the origin to P (at fixed values of t, θ and ϕ) and a spacetime curve of an observer who falls into the black hole, the proper distance for small r is given by

$$d(r) = \frac{2}{3} \frac{1}{\sqrt{2 G_0 M}} r^{3/2} \ . \tag{5.19}$$

Comparing eqs.(5.18, 5.19) valid for small r, one can obtain the value of $\gamma = \frac{9}{2}$. We shall study the black hole thermodynamics taking the general value of γ. We shall also compare the results for two different values of $\gamma = \frac{9}{2},\ 0$.

Next using the eq.(5.16), the position dependent Newton's constant takes the form

$$
\begin{aligned}
G(r) &= \frac{G_0 d(r)^2}{d(r)^2 + \tilde{\omega} G_0} \\
&= \frac{G_0 r^3}{r^3 + \tilde{\omega} G_0 (r + \gamma G_0 M)}
\end{aligned}
\tag{5.20}
$$

where $\tilde{\omega} = \omega \zeta^2$. Here the constants γ and $\tilde{\omega}$ are coming from the proper cut-off identification of the infrared momentum scale k in this formalism. Hence, the final form quantum corrected lapse function reads

$$
f(r) = 1 - \frac{2G_0 M r^2}{r^3 + \tilde{\omega} G_0 (r + \gamma G_0 M)} \ . \tag{5.21}
$$

We now first proceed to obtain the horizon of the quantum corrected Schwarzschild black hole. To determine the horizon, we write down $f(r)$ in the following form

$$
f(r) = \frac{B(x)}{B(x) + 2x^2} \tag{5.22}
$$

where $x \equiv \frac{r}{G_0 M}$. The polynomial $B(x)$ is given by

$$
B(x) \equiv B_{\gamma,\Omega}(x) = x^3 - 2x^2 + \Omega x + \gamma \Omega \tag{5.23}
$$

where

$$
\Omega = \frac{\tilde{\omega}}{G_0 M^2} \ . \tag{5.24}
$$

As Ω carries the signature of the quantum corrections, the classical limit can be recovered setting $\Omega = 0$. The horizon radius for the above is given by solving $f(r) = 0$ which in turn implies that $B(x) = 0$. For $\Omega = 0$, the non-trivial classical horizon is given by $x_0 = 2$ which corresponds to the known Schwarzschild horizon $r_0 = x_0 G_0 M = 2G_0 M$. Now in the quantum case $\Omega \neq 0$, and then $B_{\gamma,\Omega}(x)$ is a cubic polynomial which has either

one or three real roots. One can see that $B(-\infty) = -\infty$ and $B(0) = \gamma\Omega > 0$ which infers that $B(x)$ always has at least one zero on the negative real axis. The derivative $B'(x) = 3x^2 - 4x + \Omega$ is positive for $x < 0$ which implies that $B(x)$ is monotonically increasing. Hence in the negative real axis, only one root is possible for $B(x)$. Therefore, $B(x)$ has either two real roots or no roots possible on the positive real axis.
To proceed further, we define x as $x = y + \frac{2}{3}$. This leads to

$$y^3 + Py + Q = 0 \tag{5.25}$$

where $P = \Omega - \frac{4}{3}$ and $Q = \gamma\Omega + \frac{2\Omega}{3} - \frac{16}{27}$. To solve the above cubic equation for y, we employ Cardano's method. We first substitute $y = u + v$ in eq.(5.25) and get

$$y^3 - 3uvy - (u^3 + v^3) = 0 \ . \tag{5.26}$$

Comparing eqs.(5.25, 5.26), we obtain the following algebra identity

$$P = -3uv, \qquad Q = -(u^3 + v^3). \tag{5.27}$$

Substituting $u = -\frac{P}{3v}$ in Q, we obtain the following quadratic equation for b^3

$$(v^3)^2 + Qv^3 - (\frac{P}{3})^3 = 0. \tag{5.28}$$

Now solving the above equation, the solutions of v^3 and u^3 read

$$\begin{aligned} v^3 &= -\frac{Q}{2} - \sqrt{(\frac{Q}{2})^2 + (\frac{P}{3})^3} = -\frac{Q}{2} - \sqrt{\Delta} \\ u^3 &= -\frac{Q}{2} + \sqrt{(\frac{Q}{2})^2 + (\frac{P}{3})^3} = -\frac{Q}{2} + \sqrt{\Delta} \quad . \end{aligned} \tag{5.29}$$

The signs before the radicals have been chosen so that the algebraic identity $Q = -(u^3+v^3)$ holds.

We now define Δ as

$$\Delta = (\frac{Q}{2})^2 + (\frac{P}{3})^3 = \frac{1}{108}\left(27Q^2 + 4P^3\right) = \frac{1}{108}D_{\gamma,\Omega} \tag{5.30}$$

where the discriminant $D_{\gamma,\Omega}$ can be written in this form

$$D_{\gamma,\Omega} = \frac{4}{27}\left[\left(9\Omega + \frac{27}{2}\gamma\Omega - 8\right)^2 + (3\Omega - 4)^3\right] \ . \tag{5.31}$$

Depending on the value of the discriminant, the cubic roots are distinguished as real or complex. If $D_{\gamma,\Omega} < 0$, there are two real roots, for $D_{\gamma,\Omega} > 0$ no real roots exist on the positive real axis. For $D_{\gamma,\Omega} = 0$, we have a double zero on the real axis. To proceed further, we observe that the discriminant can be factorized in the following form

$$D_{\gamma,\Omega} = \Omega\left[\Omega - \Omega_1(\gamma)\right]\left[\Omega - \Omega_{cr}(\gamma)\right] \tag{5.32}$$

where

$$\begin{aligned} \Omega_1(\gamma) &= \frac{1}{2} - \frac{27}{8}\gamma^2 - \frac{9}{2}\gamma - \frac{1}{8}\left(9\gamma+2\right)^{\frac{3}{2}}\sqrt{\gamma+2} \\ \Omega_{cr}(\gamma) &= \frac{1}{2} - \frac{27}{8}\gamma^2 - \frac{9}{2}\gamma + \frac{1}{8}\left(9\gamma+2\right)^{\frac{3}{2}}\sqrt{\gamma+2} \ . \end{aligned} \tag{5.33}$$

Now the function $\Omega_1(\gamma)$ is always negative for any positive value of γ. So the sign of the discriminant solely depends on the value of $\Omega_{cr}(\gamma)$. If $\Omega < \Omega_{cr}(\gamma)$, then $D_{\gamma,\Omega} < 0$ so that $B_{\gamma,\Omega}(x)$ has two real roots namely x_+ and x_- on the real positive axis. For $\Omega = \Omega_{cr}(\gamma)$, the two roots overlap into a single root at $x_+ = x_- = x_{cr}$. For $\Omega > \Omega_{cr}(\gamma)$, there are no roots of $B_{\gamma,\Omega}(x)$ in the real positive axis.

Now for the horizon to exist, we therefore require to take $\Omega < \Omega_{cr}(\gamma)$ for which the two zeros on the real positive axis take the analytical forms

$$x_+ = y_+ + \frac{2}{3} = 2\sqrt{-\left(\frac{P}{3}\right)}\cos\left(\frac{\Theta}{3}\right) + \frac{2}{3}$$
$$x_- = y_- + \frac{2}{3} = 2\sqrt{-\left(\frac{P}{3}\right)}\cos\left(\frac{\Theta}{3} + \frac{4\pi}{3}\right) + \frac{2}{3} \tag{5.34}$$

where $\Theta = \arccos\left(-\frac{\frac{Q}{2}}{\sqrt{-(\frac{P}{3})^3}}\right)$. Substituting P and Q in terms of γ and Ω in $x_\pm$, we obtain an outer horizon r_+ and an inner horizon r_-

$$r_+ = x_+ G_0 M = \frac{2}{3}\left(1 + \sqrt{4 - 3\Omega}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{16 - 9(3\gamma + 2)\Omega}{2\sqrt{-(3\Omega - 4)^3}}\right)\right)\right) G_0 M \tag{5.35}$$
$$r_- = x_- G_0 M = \frac{2}{3}\left(1 - \sqrt{4 - 3\Omega}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{16 - 9(3\gamma + 2)\Omega}{2\sqrt{-(3\Omega - 4)^3}}\right) + \frac{\pi}{3}\right)\right) G_0 M\ . \tag{5.36}$$

These solutions are for general positive γ for which $\Omega < \Omega_{cr}(\gamma)$. We shall work with $\gamma = \frac{9}{2}$, for which the critical value is calculated from eq. (5.33) to be $\Omega_{cr} \approx 0.20$. Now taking the limit $\Omega \to 0$ in the above solution, we obtain the classical horizon where $r_+ = 2G_0M$ and $r_- = 0$. For $\gamma = 0$, we have $x_\pm = 1$ for $\Omega = \Omega_{cr} = 1$. From the definition of Ω, the critical value for the mass is calculated in terms of $\Omega_{cr}(\gamma)$, which reads

$$M_{cr}(\gamma) = \left[\frac{\tilde{\omega}}{\Omega_{cr}(\gamma) G_0}\right]^{1/2}\ . \tag{5.37}$$

For $\Omega = \Omega_{cr}$, we have a double zero of $f(r)$ on the positive real axis where $r_\pm$ coincides at $r_{cr} = x_{cr} G_0 M$. The explicit solution for r_{cr} is given in [111].

5.3.1 Large mass expansion of r_+

We are mainly interested on the outer horizon r_+ to study the black hole thermodynamics. We shall carry out a large mass expansion of x_+ upto $\mathcal{O}(\Omega^2)$ which in turn would give the outer horizon r_+. Starting with the classical horizon $r_+ = 2G_0M$, that is, $x_+ = 2$, the ansatz can be taken in the form [111]

$$x_+ = 2 + c_1\Omega + c_2\Omega^2 + \mathcal{O}(\Omega^3) \ . \tag{5.38}$$

Substituting this form of ansatz for x_+ in $B_{\gamma,\Omega}(x)$ and comparing equal powers of Ω and Ω^2, we can obtain c_1 and c_2 in the following form

$$c_1 = -\frac{1}{4}(2+\gamma), \quad c_2 = -c_1^2 - \frac{c_1}{4} = -\frac{(2+\gamma)(1+\gamma)}{16} \ . \tag{5.39}$$

Putting c_1 and c_2 in eq.(5.38), we get

$$x_+ = \left(2 - \frac{1}{4}(2+\gamma)\,\Omega - \frac{1}{16}(2+\gamma)(1+\gamma)\Omega^2 + \mathcal{O}(\Omega^3)\right) \ . \tag{5.40}$$

Substituting Ω in terms of the black hole mass M, the quantum corrected outer horizon r_+ in the large mass expansion reads

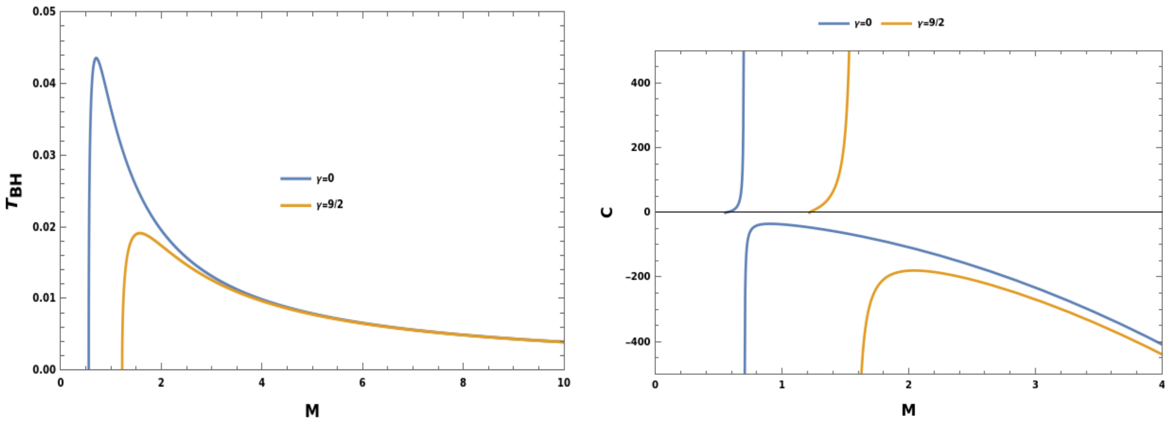
$$r_+ = x_+G_0M = 2G_0M - \frac{(2+\gamma)\,\tilde{\omega}}{4M} - \frac{(2+\gamma)(1+\gamma)\tilde{\omega}^2}{16G_0M^3} + \mathcal{O}(\frac{1}{G_0^2M^5}) \ . \tag{5.41}$$

Note that the quantum corrected horizon r_+ is smaller than the classical horizon.

5.4 Heat capacity and entropy of the black hole

In this section, we will explore the thermodynamic quantities such as heat capacity and entropy with general positive γ in case of the quantum corrected Schwarzschild geometry. For this we start with the general Bekenstein-Hawking temperature of black holes which reads

$$T_{BH} = \frac{1}{4\pi}f'(r_+) \ . \tag{5.42}$$



(a) T_{BH} vs M (b) C vs M

Fig. 5.1 The quantum corrected Hawking temperature and specific heat vs mass of the black hole in Planck unit.

Differentiating $f(r)$ from eq.(5.21) at the outer horizon r_+, the quantum corrected Hawking temperature is given by [111]

$$T_{BH} = \frac{1}{8\pi G_0 M}\left[1 - \frac{\Omega}{x_+^2} - \frac{2\gamma\Omega}{x_+^3}\right] . \tag{5.43}$$

Now for $\gamma = 0$ and $\Omega = \Omega_{cr} = 1$, the black hole temperature is zero [111]. We also observe that at $\Omega = \Omega_{cr} = 1$, the heat capacity also vanishes for $\gamma = 0$. Using x_+ from eq.(5.35) and converting Ω in terms of the mass of the black hole M, and using eq.(5.43), one can obtain the Hawking temperature. The quantum corrected Hawking temperature is plotted with the mass of the black hole M taking $\tilde{\omega} = 0.3$ in Fig. (5.1a) for $\gamma = 0$ and $\gamma = \frac{9}{2}$. Note that we have set $G_0 = 1$ throughout all the plots. From (5.1a), we can infer that for large mass black holes, the temperature almost coincides for $\gamma = 0$ and $\gamma = 9/2$ but when the mass of the black hole is small the temperature is relatively low for $\gamma = \frac{9}{2}$ than $\gamma = 0$. Another feature that is worthy to note is that the temperature of the black hole vanishes for higher mass for $\gamma = \frac{9}{2}$ than $\gamma = 0$. So the black hole critical mass or the remnant mass is large for $\gamma = \frac{9}{2}$. This can also be seen analytically by solving two equations. The first one follows from the vanishing of the Hawking temperature of the black hole which takes zero value as it goes to critical mass and it reads

$$x_{cr}^3 - \Omega_{cr}x_{cr} - 2\gamma\Omega_{cr} = 0 \ . \tag{5.44}$$

The second one is more general and follows from the vanishing of the lapse function at the event horizon which from eq.(5.23) reads

$$x_{cr}^3 - 2x_{cr}^2 + \Omega_{cr}x_{cr} + \gamma\Omega_{cr} = 0 \ . \tag{5.45}$$

Combining these two equations, we get

$$2x_{cr}^2 - 2\Omega_{cr}x_{cr} - 3\gamma\Omega_{cr} = 0 \ . \tag{5.46}$$

Solving this, we get the critical radius of the black hole in terms of Ω_{cr} and γ in the form

$$x_{cr} = \frac{\Omega_{cr}}{2}\left[1 \pm \sqrt{1 + \frac{6\gamma}{\Omega_{cr}}}\right] \ . \tag{5.47}$$

As the horizon radius $r_{cr} = x_{cr}G_0M$ which corresponds to double zero of $f(r)$ is always positive, x_{cr} should always be positive for all values for γ. Hence one needs to take the positive sign before the square root in eq.(5.47). Now substituting $x_{cr} = \frac{\Omega_{cr}}{2}\left[1 + \sqrt{1 + \frac{6\gamma}{\Omega_{cr}}}\right]$ in eq. (5.44), we get exactly the same value of Ω_{cr} in terms of γ as in eq. (5.33). Substituting eq.(5.40) in the exact form of Hawking temperature (eq.(5.43)), the large mass expansion of T_{BH} reads upto $\mathcal{O}(M^{-4})$

$$\begin{aligned} T_{BH} &= \frac{1}{8\pi G_0 M}\left[1 - \frac{\Omega}{4}(1+\gamma) - \frac{\Omega^2}{32}(2+\gamma)(2+3\gamma) + \mathcal{O}(\Omega^3)\right] \\ &= \frac{1}{8\pi G_0 M}\left[1 - \frac{\Omega_{cr}}{4}(1+\gamma)\left(\frac{M_{cr}}{M}\right)^2 - \frac{\Omega_{cr}^2}{32}(2+\gamma)(2+3\gamma)\left(\frac{M_{cr}}{M}\right)^4 + \mathcal{O}(M^{-6})\right] \end{aligned} \tag{5.48}$$

where in the second line of the equality we have used $\Omega = \Omega_{cr}\left(\frac{M_{cr}}{M}\right)^2$. The above expression for the temperature for large mass holds for any value of γ. It is also reassuring to note that eq.(5.48) reduces to the appropriate expression in the limit $\gamma \to 0$ [111].

Next we define the heat capacity of the black hole which can be written identifying the internal energy of the black hole with the its total mass M

$$C = \frac{\mathrm{d}M}{\mathrm{d}T_{BH}} = \left(\frac{\mathrm{d}T_{BH}}{\mathrm{d}M}\right)^{-1} = \left(\frac{\mathrm{d}T_{BH}}{\mathrm{d}\Omega}\frac{\mathrm{d}\Omega}{\mathrm{d}M}\right)^{-1} . \tag{5.49}$$

Using eq.(5.43), the heat capacity can be written in terms of $x_+(\Omega)$

$$C = -\frac{8\pi G_0 M^2 x_+^4}{[4\Omega^2(3\gamma + x_+)x'_+ + x_+\left(x_+^3 - 3\Omega x_+ - 6\gamma\Omega\right)]} \tag{5.50}$$

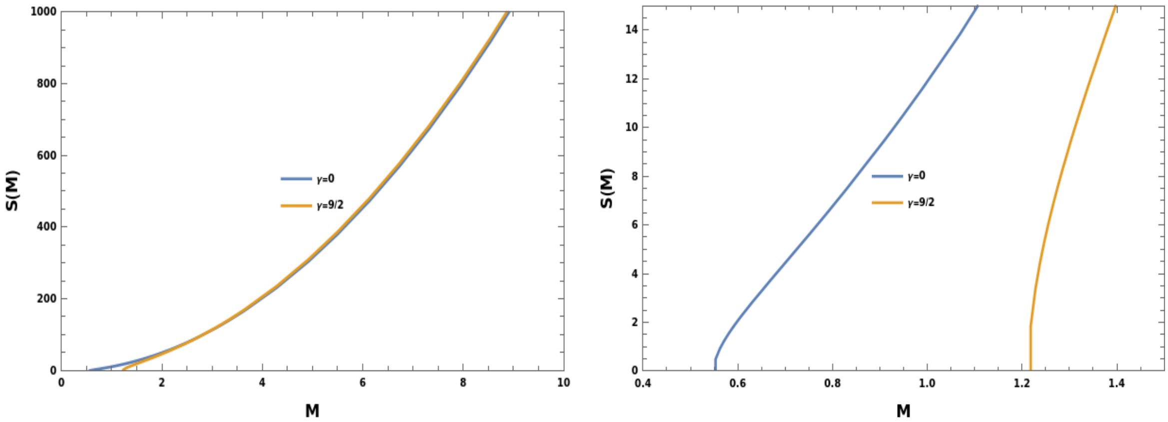
where prime represents derivative with respect to Ω. Next substituting x_+ in terms of γ and Ω from eq.(5.35) in eq.(5.50) where Ω is defined in eq.(5.24), we have plotted the specific heat capacity against the mass of the balck hole for $\gamma = 0, \frac{9}{2}$. In Fig. (5.1b), the mass where the specific heat diverges is the same mass where $\frac{\mathrm{d}T_{BH}}{\mathrm{d}M}(\tilde{M}_{cr}) = 0$ in Fig. (5.1a). The specific heat is negative for $M > \tilde{M}_{cr}$ and goes to positive specific heat between $M_{cr} < M < \tilde{M}_{cr}$. Another point to note is that at critical mass M_{cr}, the specific heat also vanishes for all γ just like the black hole temperature.

The specific heat for large mass black hole for any value of γ reads

$$C = -8\pi G_0 M^2\left[1 + \frac{3}{4}(1+\gamma)\Omega + \left(\frac{5}{32}(2+\gamma)(2+3\gamma) + \frac{9}{16}(1+\gamma)^2\right)\Omega^2 + \mathcal{O}(\Omega^3)\right] . \tag{5.51}$$

In the limit $M \to \infty$, the specific heat approaches the classical value $-8\pi G_0 M^2$ which matches with the specific heat in the limit $\gamma \to 0$ [111].

We now proceed to calculate the entropy from the general law of thermodynamics $\left(\frac{\partial S}{\partial U}\right)_V = \frac{1}{T}$. In the context of black hole thermodynamics, U is identified as the mass of the balck hole M. The differential form of the black hole entropy is then given by $\frac{\mathrm{d}S}{\mathrm{d}M} = \frac{1}{T_{BH}}$.



(a) $S(M)$ vs. M (b) $S(M)$ vs. M near M_{cr}

Fig. 5.2 The entropy $S(M)$ vs. the black hole mass M for $\gamma = 0, \frac{9}{2}$ with $\tilde{\omega} = 0.3$.

This leads to

$$S = \int \frac{\mathrm{d}M}{T_{BH}} \ . \tag{5.52}$$

Using x_+ in terms of the black hole mass from eq.(5.35) and the black hole temperature T_{BH} from eq.(5.43) in the above equation and solving it numerically, we obtain the result plotted in Fig. (5.2). Next we calculate the black hole entropy for the large mass black hole using the temperature from eq. (5.48) in the form

$$S = \int \Bigg[8\pi G_0 M + 2\pi\tilde{\omega}\frac{(1+\gamma)}{M} + \frac{\tilde{\omega}^2\pi}{4G_0M^3}\Big((2+\gamma)(2+3\gamma) + 2(1+\gamma)^2\Big) + \mathcal{O}(M^{-5})\Bigg] \mathrm{d}M \ . \tag{5.53}$$

Integrating the above equation, the entropy upto $\mathcal{O}(M^{-4})$ reads

$$S = 4\pi G_0 M^2 + 2\pi\tilde{\omega}(1+\gamma)\ln M - \frac{\tilde{\omega}^2\pi}{8G_0M^2}\Big((2+\gamma)(2+3\gamma) + 2(1+\gamma)^2\Big) + \mathcal{O}(M^{-4}) + S_0 \tag{5.54}$$

where S_0 is the constant of integration.

We can determine the integration constant S_0 by fixing $S \to 0$ if $M \to M_{cr}$ since $C \to 0$.

Then S_0 gets evaluated to

$$S_0 = -4\pi G_0 M_{cr}^2 - 2\pi\tilde{\omega}(1+\gamma)\ln M_{cr} + \frac{\tilde{\omega}^2\pi}{8G_0 M_{cr}^2}\left((2+\gamma)(2+3\gamma)+ 2(1+\gamma)^2\right) + \mathcal{O}(M_{cr}^{-4}) \ . \tag{5.55}$$

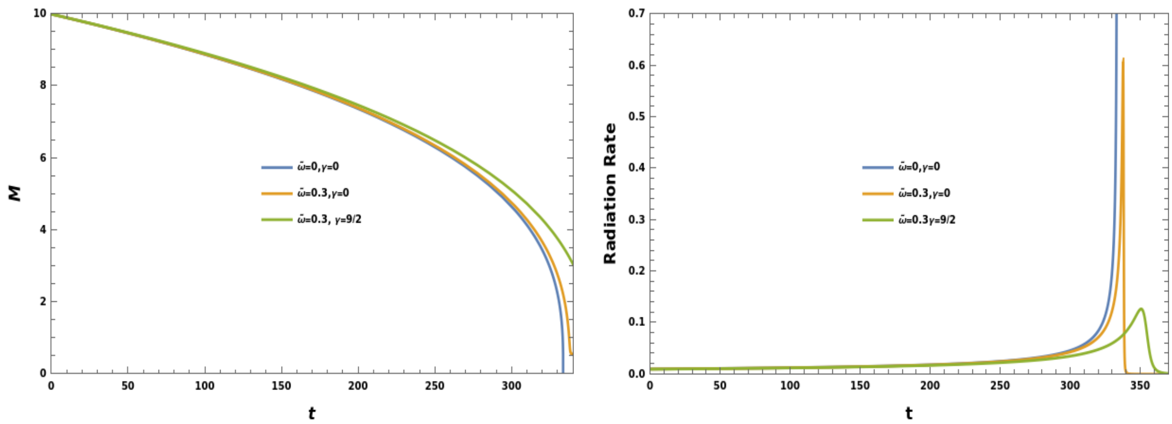
In terms of the horizon area of the black hole, the expression for the entropy takes the form

$$S = \frac{A_{class}}{4G_0} + \pi\tilde{\omega}(1+\gamma)\ln\left(\frac{A_{class}}{16\pi G_0^2}\right) - \frac{2\tilde{\omega}^2\pi G_0}{A_{class}}\left((2+\gamma)(2+3\gamma)+2(1+\gamma)^2\right) + S_0 + \mathcal{O}(A_{class}^{-2}) \tag{5.56}$$

where A_{class} is the area defined with the classical horizon radius given by $A_{class} = 4\pi(2G_0M)^2$. The first term is the classical entropy which is recovered for the heavy mass black hole. It is to be noted that the leading and subleading quantum correction terms get affected by the value of γ. The leading quantum correction term for entropy of the black hole takes higher value for $\gamma = \frac{9}{2}$ than $\gamma = 0$.

5.5 Black hole mass evaporation with time

If the ambient temperature is smaller than the the black hole temperature, the black hole radiates off its energy increasing the temperature. As a result, the mass of the black hole decreases for that case. Now from Fig. (5.1), one can see that the qualitative nature of the Hawking temperature for large mass for $\gamma = 0, \frac{9}{2}$ is similar with the classical behaviour. When the mass reaches as small as $\tilde{M}_{cr}$, the quantum corrected temperature takes the maximum value. After $\tilde{M}_{cr}$, the temperature quickly reaches zero for mass M_{cr}. Now the massive quantum corrected black hole becomes continuously evaporating increasing its temperature during the process. But the Hawking temperature unlike the classical case can never exceed beyond $T_{BH}(\tilde{M}_{cr})$. Now if one consider the loss of its energy due to photon radiation, the Stefan-Boltzmann law can be employed to calculate



(a) The mass of the black hole vs. time.

(b) The radiation rate vs. time

Fig. 5.3 The mass is in units of Planck mass and the time is in units of characteristic time.

the mass loss as a function of time. It reads [142]

$$-\frac{\mathrm{d}M}{\mathrm{d}t} = \sigma \mathcal{A}(M) T_{BH}^4 \tag{5.57}$$

where σ is Stefan-Boltzmann constant and $\mathcal{A}(M) = 4\pi r_+^2 = 4\pi G_0^2 M^2 x_+^2$ is the area of the outer horizon. Substituting eq(s).(5.35, 5.43) in the above equation, we can recast it in terms of $x = \frac{M}{M_{Pl}}$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{x_+^2}{4t_{ch}x^2}\left[1 - \frac{\Omega}{x_+^2} - \frac{2\gamma\Omega}{x_+^3}\right] \tag{5.58}$$

where the characteristic time is defined as $t_{ch} = \frac{\sigma}{256\pi^3 G_0^2 M_{Pl}^3}$. Solving the above differential equation numerically starting from some initial value x_i (here $x_i = 10$) upto $M \approx M_{cr}$, the mass and the radiation rate as function of time are plotted for classical case ($\gamma = 0$ and $\tilde{\omega} = 0$) and for $\gamma = 0, 9/2$ in Fig.(5.3). Here we have taken M close to M_{cr} for solving the mass as a function of time, as T_{BH} vanishes at M_{cr}. It is to be noted that in Fig. (5.3b), the radiation rate decreases after $\tilde{M}_{cr}$ and reaches to zero at M_{cr}. As a result the time diverges if the black hole mass goes to its critical value M_{cr} [111].

Next we proceed to calculate the approximate analytical result for the mass loss as a function of time employing the Hawking temperature and outer horizon radius for the large mass expansion. The rate at which the energy radiates is written upto $\mathcal{O}(\Omega)$ in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{t_{ch}x^2}\left[1 - \frac{(6+5\gamma)}{4}\frac{\tilde{\omega}}{x^2} + \mathcal{O}(\tilde{\omega}^2)\right] . \tag{5.59}$$

It is also to be noted that in the absence of the quantum effects, the mass evaporation rate with respect to time reads (using the fact $x_+ = 2$) [142, 160, 161]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{t_{ch}x^2} . \tag{5.60}$$

Solving this for initial value x_i at $t = 0$, yields the mass time relation for the classical case

$$x = \left(x_i^3 - \frac{3t}{t_{ch}}\right)^{1/3} . \tag{5.61}$$

Hence the rate at which the black hole radiates is given in terms of time as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{t_{ch}\left(x_i^3 - \frac{3t}{t_{ch}}\right)^{2/3}} . \tag{5.62}$$

The evaporation rate of mass indicates that it will stop at a time $\frac{t}{t_{ch}} = \frac{1}{3}x_i^3$ and the rate goes to infinity when the process stops for the classical black hole.

Next we proceed to carry out the above analysis for the quantum corrected black hole in the large mass expansion limit using eq.(5.59). Solving the differential equation (5.59), we get the solution for the mass of black hole upto $\mathcal{O}(\tilde{\omega})$ in the following form

$$x = \left[-\frac{3t}{t_{ch}} + x_i^3 + \frac{3(6+5\gamma)\tilde{\omega}}{4}x_i - \frac{3(6+5\gamma)\tilde{\omega}}{4}\left(x_i^3 - \frac{3t}{t_{ch}}\right)^{1/3} + \mathcal{O}(\tilde{\omega}^2)\right]^{1/3} \tag{5.63}$$

and the rate at which the black hole radiates is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{t_{ch}\left[-\frac{3t}{t_{ch}} + x_i^3 + \frac{3(6+5\gamma)\tilde{\omega}}{4}x_i - \frac{3(6+5\gamma)\tilde{\omega}}{4}\left(x_i^3 - \frac{3t}{t_{ch}}\right)^{1/3}\right]^{2/3}} \times \left[1 - \frac{(6+5\gamma)\tilde{\omega}}{4\left[-\frac{3t}{t_{ch}} + x_i^3 + \frac{3(6+5\gamma)\tilde{\omega}}{4}x_i - \frac{3(6+5\gamma)\tilde{\omega}}{4}\left(x_i^3 - \frac{3t}{t_{ch}}\right)^{1/3}\right]^{2/3}}\right] . \tag{5.64}$$

Note that this approximation is valid upto the mass $\tilde{M}_{cr}$ after which the quantum effect dominates. From the rate of radiation one can conclude that the approximate time in which the black hole evaporates is given (upto $\mathcal{O}(\tilde{\omega})$) by

$$\frac{t}{t_{ch}} = \frac{x_i^3}{3} + \frac{(6+5\gamma)\tilde{\omega}}{4}x_i \ . \tag{5.65}$$

It should be noted that the above results hold for any value of γ lying between 0 and $\frac{9}{2}$.

5.6 Free energy and phase transition

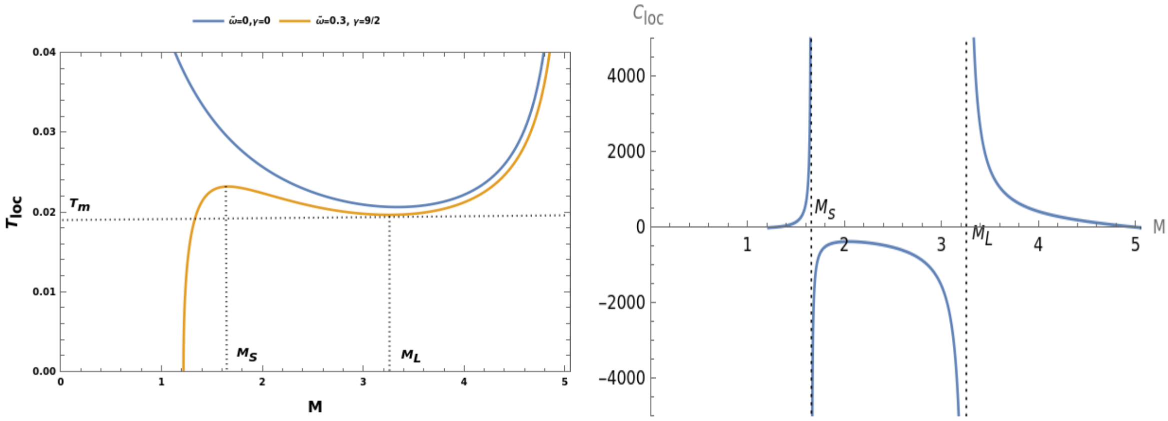
In this section, we move on to study phase transition for the quantum corrected asymptotically safe Schwarzschild black hole. It is known that stable thermodynamic equilibrium cannot be established in asymptotically flat spacetime. For this we enclose the black hole inside a finite concentric spherical cavity of radius R which is larger than the black hole radius. The black hole within the cavity acts as a well defined thermodynamic canonical ensemble. This cavity can play a similar role as the boundary of ADS space described in [162, 163]. To proceed further, we need to calculate the local black hole temperature in the cavity in order to obtain the local thermodynamic energy and heat capacity. The local thermodynamic analysis in terms of Euclidean Einstein action is proposed by York [164].

The local temperature T_{loc} is defined on the surface of the cavity. The local temperature as seen by a local observer at R is written as [164–166]

$$
\begin{aligned}
T_{loc} &= \frac{T_{BH}}{\sqrt{-g_{00}}} \\
&= \frac{\left(1 - \frac{\Omega}{x_+^2} - \frac{2\gamma\Omega}{x_+^3}\right)}{8\pi G_0 M\sqrt{1 - \frac{2G_0 M R^2}{R^3 + \tilde{\omega} G_0 (R + \gamma G_0 M)}}} \,.
\end{aligned}
\tag{5.66}
$$

In Fig. (5.4a), we have plotted the local temperature vs the mass of the black hole taking $\gamma = \frac{9}{2}$, $\tilde{\omega} = 0.3$ and for ordinary Schwarzschild black hole with $R = 10$. As $R \to \infty$, the local temperature approaches T_{BH}. Also one can observe from Fig. (5.4a) the existence of local minimum at $M = M_L$ and a local maximum at $M = M_S$. We now call this maximum temperature T_M at M_S and minimum temperature T_m at M_L. We can see from Fig. (5.4a) that for the ordinary Schwarzschild black hole (where $\gamma = 0$ and $\tilde{\omega} = 0$) inside a cavity of radius R, there exists a minimum temperature in the plot. Above this temperature, there are two values for M for a given value of temperature. The black hole solution does not exist below this minimum temperature, and the radiation state (thermal AdS) prevails there. This nature exactly matches with the SAdS black hole described in [162, 167].

The black hole geometry we are considering here exists for all possible temperatures. From the plot we can observe that one black hole state is possible below the minimum temperature T_m, we shall call it the "tiny black hole". Between $T_m < T < T_M$, there are three values of M which in turn implies that there are three horizon radii r_+. For temperature $T > T_M$, only one black hole state exists, we call it the "large black hole". Interestingly, no radiation state exists unlike the ordinary Schwarschild black hole case inside a cavity. We will again encounter these three black hole states once we calculate the specific heat capacity. Now it is well known that in Hawking-Page phase transition which occurs in SAdS black hole, if we increase the temperature, there is a phase transition from the thermal AdS spacetime (radiation state) to the SAdS black hole after a certain minimum temperature [162, 167]. In our case, if we increase the temperature, there is a phase transition from one black hole state to another. So for all temperatures, the black



(a) The local temperature vs the mass of the black hole.

(b) The local specific heat vs. the mass of the black hole ($\gamma = \frac{9}{2}$)

Fig. 5.4 T_{loc} and C_{loc} with mass of the black hole for $\tilde{\omega} = 0.3$ and $R = 10$.

hole state prevails instead of the radiation state. It means there is no Hawking-Page phase transition here. We will again confirm this observation from the free energy calculation. Next, using the first law of thermodynamics $\mathrm{d}E = T\mathrm{d}S$, one can write down the local thermodynamic energy as

$$\begin{aligned} E_{loc} &= \int_{M_{cr}}^{M} T_{loc}\mathrm{d}S \\ &= \int_{M_{cr}}^{M} \frac{T_{loc}}{T_{BH}}\mathrm{d}M \\ &= \int_{M_{cr}}^{M} \frac{\sqrt{R^3 + \tilde{\omega}G_0(R+\gamma G_0 M)}}{\sqrt{R^3 - 2G_0MR^2 + \tilde{\omega}G_0(R+\gamma G_0 M)}}\mathrm{d}M \end{aligned} \tag{5.67}$$

where we have used eqs.(5.52, 5.66). Integrating the above expression, we obtain the form of E_{loc} to be

$$\begin{aligned} E_{loc} = &\frac{\sqrt{R^3+\tilde{\omega}G_0(R+\gamma G_0 M)}\sqrt{R^3-2G_0MR^2+\tilde{\omega}G_0(R+\gamma G_0 M)}}{\left(\tilde{\omega}\gamma G_0^2-2G_0R^2\right)} - \frac{2G_0R^2\left(R^3+\tilde{\omega}G_0R\right)}{\sqrt{\tilde{\omega}\gamma G_0^2}\left(\tilde{\omega}\gamma G_0^2-2G_0R^2\right)^{3/2}} \\ &\times \sinh^{-1}\left(\frac{\sqrt{\tilde{\omega}\gamma G_0^2-2G_0R^2}\sqrt{R^3+\tilde{\omega}G_0(R+\gamma G_0 M)}}{\sqrt{2G_0R^2(R^3+\tilde{\omega}G_0R)}}\right) - E_{loc}(M_{cr}) \ . \end{aligned} \tag{5.68}$$

Note that E_{loc} goes to $E_{loc} = M - M_{cr}$ in the limit $R \to \infty$. This condition is expected physically since for the quantum corrected black hole, there would be a critical mass, and therefore the amount of energy that can be extracted out of the black hole would be

$(M - M_{cr})$. Now for $\tilde{\omega} = 0$, the local energy goes to $E_{loc} = M$ in the limit $R \to \infty$ as $M_{cr} = 0$.

To investigate the thermodynamic stability of the quantum corrected black hole, we now calculate the local heat capacity C_{loc}. This reads

$$C_{loc} = \frac{\partial E_{loc}}{\partial T_{loc}} = \frac{\partial E_{loc}}{\partial M}\frac{\partial M}{\partial T_{loc}} \tag{5.69}$$

In Fig. (5.4b), the local specific heat is plotted as a function of M. It is well known that the positive heat capacity ensures that the thermodynamic canonical system is in thermal stability. But for the negative heat capacity the black hole radiates. In Fig.(5.4b), we can observe three regions separated by discontinuity. The discontinuity or divergent point corresponds to the extrema of the local temperature. For $M > M_L$, the specific heat is positive which ensures that the balck hole is thermally stable within the cavity. We call it large stable black hole (LSB). Between $M_S < M < M_L$, the negative specific heat makes sure that the black hole will be in unstable state. This is called the small unstable black hole (SUB). Another region is between $M_{cr} < M < M_s$ where the local specific heat is again positive ensuring its thermal stability. We call it tiny stable black hole(TSB). It is to be noted that the heat capacity vanishes at non-zero value of M which is identical to the critical value M_{cr} calculated earlier.

With the above results in hand, we are now in a position to study the phase transition of the quantum corrected Schwarzschild black hole. For that we calculate the on-shell free energy of the black hole [164, 168]

$$F_{on} = E_{loc} - T_{loc}S \tag{5.70}$$

where E_{loc}, T_{loc} are given in eqs.(5.66, 5.68)respectively. To calculate the free energy, We have used the numerical solution of the entropy S using the eq.(5.52). It is plotted in the Fig. (5.2).

In Fig. (5.5), we have shown F_{on} curves with T_{loc} for $\gamma = \frac{9}{2}$ using $\tilde{\omega} = 0.3$ and $R = 10$. Before proceeding further, we need to define the radiation state or hot curved space

(HCS). The on-shell free energy vanishes for radiation state ($F_{on} = 0$) for any arbitrary T_{loc}. This occurs as $E_{loc} = 0$, $S = 0$ for radiation state. In Fig. (5.5), the T_{loc} axis represents the on-shell free energy for radiation state as $F_{on} = 0$ on this axis. From the plot we observe that the on-shell free energy of the radiation state is always larger than the on-shell free energy of the black hole state. Hence, the radiation state is never realizable.

Next we move on to discuss the phase transition between the three states of black hole, namely, LSB, SUB and TSB by studying their free energy. From Fig. (5.5), one can observe that there are two critical points, namely, T_{c1} and T_{c2} corresponding to the phase transition from one black hole state to another.

When the temperature of the black hole (T_{loc}) is less than T_m, that is, $0 \leq T_{loc} < T_m$, the on-shell free energy of the tiny stable black hole F_{on}^{TSB} is negative. In this region only one black hole state, that is, TSB exists. Between $T_m \leq T_{loc} < T_{c1}$, we have $F_{on}^{SUB} < F_{on}^{LSB} < F_{on}^{TSB}$. It implies that the small, large black hole eventually decay into the tiny black hole state when the temperature of the black hole is less than the critical temperature T_{c1}. For $T_{c1} < T_{loc}$, the on-shell free energy of large black hole is less than any of the state. Hence, one can conclude that below the critical temperature T_{c1} the tiny black hole state (TSB) is more probable while above T_{c1} the large black state is the thermally stable state. Note that the quantum corrected black hole exhibits a first order phase transition between tiny black hole to large black hole state since the first derivative of the free energy does not exist at the transition point. In the temperature region between $T_{c1} \leq T_{loc} < T_{c2}$ where the three black hole exist, the on-shell free energy satisfies $F_{on}^{SUB} < F_{on}^{TSB} < F_{on}^{LSB}$. Here the small black hole has positive free energy so it decays into either the TSB or LSB black hole state. At T_{c2}, the small black hole has zero free energy value. When $T_{c2} \leq T_{loc} < T_M$, all three black hole states are in stable state because all three have negative free energy. Though, the on-shell free energy of the SUB and TSB are still larger than the LSB. Therefore, they undergo tunneling and eventually decay into the LSB. Finally, for $T > T_M$, there is only one state of the black hole possible, namely, the LSB.

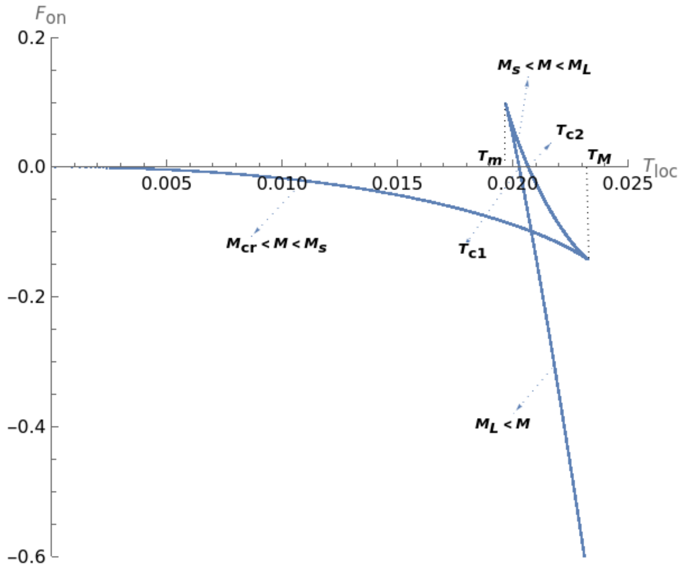


Fig. 5.5 On-shell free energy vs. the local temperature for $\gamma = \frac{9}{2}$, $\tilde{\omega} = 0.3$ and $R = 10$.

Note that for ordinary Schwarzschild black hole [169] like SAds black hole [162], Hawking-Page phase transition is possible which means that the radiation state prevails below a critical temperature while the large black hole state is more probable than the radiation state above this critical temperature. Remarkably, in quantum corrected Schwarzschild black hole, there is no Hawking-Page phase transition as the black hole state always prevails rather than a radition state.

5.7 Conclusions

In this chapter, we investigate the thermodynamics and phase transition of the renormalized group improved asymptotically safe Schwarzchild black hole. The method of renormalization group improvement for gravitational constant has been used in order to take into account the quantum gravitational effect into the spacetime. Here we first have derived the analytical solution of the outer horizon and inner horizon $r_{\pm}$ using $f(r) = 0$ for general γ depending on the value of the discriminant. From the large mass expansion, we have observed that the event horizon is smaller than the classical event horizon.
Next we have computed the exact result for the quantum corrected Hawking temperature, specific heat and plotted it for $\gamma = 0, \frac{9}{2}$. Here we have seen that upto $\tilde{M}_{cr}$, the Hawking

temperature obeys the semi classical $1/M$ laws while the quantum effects are prominent after the mass $\tilde{M}_{cr}$ where the black hole temperature takes maximum value and reaches $T_{BH} = 0$ at $M = M_{cr}$ (the critical or remnant mass). The specific heat diverges at $M = \tilde{M}_{cr}$ and gets negative for $M > \tilde{M}_{cr}$ but positive in between $M_{cr} < M < \tilde{M}_{cr}$. We have also calculated the Hawking temperature and specific heat for the large mass expansion for general γ. We then numerically computed and plotted the entropy of the black hole for $\gamma = \frac{9}{2}, 0$ and tried to report the differences in results for the different values of γ. The area theorem is recovered along with the logarithmic correction. Interestingly, the leading quantum correction term, that is, the logarithmic term gets bigger in the presence of γ. We further calculated and plotted the mass output and radiation rate with numerically solving the mass loss equation. We have noticed after $\tilde{M}_{cr}$, the radiation rate decreases and vanishes at the critical mass. The infinitely distant observer sees that the black hole reached its final state after an infinite time. We have also observed a analytical picture of radiation rate and mass output upto $\tilde{\omega}$ order until the mass reaches $\tilde{M}_{cr}$.

We then moved on to investigate the phase transition and thermodynamic stability of the black hole introducing the concept of a local observer. To do this, we first calculate the local temperature, specific heat and the on-shell free energy for the quantum corrected Schwarzschild black hole. It is observed from the specific heat that three possible states of the black hole exist, namely, the large stable black hole (LSB), small unstable black hole (SUB) and the tiny stable black hole (TSB). Here we also observed from the on-shell free energy that the black hole state is always more stable than the radiation state. So it is interesting to note that in quantum corrected Schwarzschild black hole, no Hawking-Page transition takes place unlike the ordinary Schwarzschild black hole. It has been also observed that there are two critical points, namely, T_{c1} and T_{c2} corresponding to the phase transition from one black hole state to another.

Chapter 6

HBAR entropy of an atom falling into a quantum corrected black hole

6.1 Introduction

The fact that radiation is emitted when atoms fall into a black hole has been an important area of research in recent times [71]. The foundations of this problem lie in the general theory of relativity known to be the brain child of Albert Einstein [170]. The amazing finding in the atoms falling into a black hole problem is that the acceleration radiation emitted, despite having a different form to a distant observer, resembles Hawking radiation. The entropy of this radiation has been termed as horizon brightened acceleration radiation (HBAR) entropy [71]. Indeed, such a result throws light on the connection between black hole physics and quantum optics [71, 171, 172]. The study also gave an insight into the principle of equivalence in general relativity. Historically, this connection was preceded by the link between gravitation, geometry and thermodynamics. The formulation of black hole thermodynamics [4–7], discovery of Hawking radiation [6, 7], particle emission from black holes [173–175], the Unruh effect [64], and acceleration radiation [65, 176–185] reinforces this deep connection.

Quantum mechanics on the other hand has been another pillar of theoretical physics. However, its unification with general relativity has proved to be a notorious problem. One of the main directions of research in theoretical physics has been to search for a unified theory of quantum gravity [186–189]. An alternative approach to quantum gravity based on the Asymptotic Safety (AS) idea has also attached a lot of interest in the recent past. The formalism of AS in gravity is based on a scale dependent Effective Average Action (EAA), which is covered in great length in chapter 2. EAA describes all gravitational phenomena taking into account the effect of all loops [26, 47, 99, 111]. As a function of this scale, the effective average action satisfies a renormalization group equation. This then results in the flow of the Newton's gravitational constant as a function of the scale. An effective improved black hole solution was then obtained from the flow of the Newton's gravitational constant, whose metric reads [111]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{6.1}$$

where

$$f(r) = 1 - \frac{2G(r)M}{r} \tag{6.2}$$

and $G(r)$ in natural units reads

$$G(r) = \frac{G_0}{1 + \frac{\tilde{\omega} G_0}{r^2}} \tag{6.3}$$

where $\tilde{\omega}$ is a constant that involves the quantum gravity correction to the black hole geometry coming from the renormalization group approach. (See section (5.3) of chapter 5 for a detailed discussion of this metric.) To simplify the algebra and get a clear result, we utilised $\gamma = 0$ in eq.(5.20) to obtain equation (6.3). Our plan in this work is to consider atoms falling into this black hole which incorporates quantum gravity corrections and draw conclusions from such a thought experiment. The aim is to look at two important physical issues. First of all, we would like to get an equivalent insight on the Einstein's principle of equivalence. There were several attempts to provide an alternative explanation of the equivalence principle [72, 190–195]. Such an insight was obtained earlier in [193, 195] where the spontaneous excitation of a two-level atom in presence of a perfectly reflecting mirror in the generalized uncertainty principle framework was studied. It was shown in [195] that when the mirror is accelerating, the spatial oscillation of the probability of excitation of the atom gets modulated due to the generalized uncertainty principle, thereby leading to an explicit violation of the equivalence principle. Now atom falling into a black hole emits radiation which has exactly the same thermal spectrum, as that radiated by a fixed atom near an accelerating mirror. This can be then thought of as a different manifestation of the equivalence principle than the standard elevator description. In this chapter (based on the work [46]), our goal is to look at the status of this principle when an atom falls into a renormalization group "improved" black hole spacetime. The second aim is to look for quantum gravity corrections in the HBAR entropy and see whether they are logarithmic in nature similar to the corrections in the Bekenstein-Hawking entropy [4–7, 196].

6.2 The quantum corrected black hole metric in Rindler form

To begin our analysis, we start from the Minkowski metric in 1+1-dimensional form

$$ds^2 = c^2 dt^2 - dz^2 \ . \tag{6.4}$$

For a particle moving with a constant proper acceleration 'a' in this flat spacetime, the time and position coordinates in terms of the proper time τ can be written as

$$t(\tau) = \frac{c}{a} \sinh\left(\frac{a\tau}{c}\right) \tag{6.5}$$

$$z(\tau) = \frac{c^2}{a} \cosh\left(\frac{a\tau}{c}\right) \ . \tag{6.6}$$

We shall now use the following coordinate transformations in eq.(6.4)

$$t = \frac{\varrho}{c}\sinh(\frac{\tilde{a}\tilde{t}}{c}) \tag{6.7}$$

$$z = \varrho\cosh(\frac{\tilde{a}\tilde{t}}{c}) \ . \tag{6.8}$$

Using eq.(s)(6.7,6.8) in eq.(6.4), the form of the metric gets modified as

$$ds^2 = \left(\frac{\tilde{a}\varrho}{c^2}\right)^2 c^2 d\tilde{t}^2 - d\varrho^2 \ . \tag{6.9}$$

Eq.(6.9) is said to be the Rindler form of the metric describing constant acceleration of a particle. If we now compare eq.(6.7) with eq.(6.5), we see that the proper time of the particle takes the form

$$\tau = \frac{\tilde{a}\tilde{t}}{a} \ . \tag{6.10}$$

Comparing eq.(6.8) with eq.(6.6), we get the form of the uniform acceleration of the particle in Minkowski spacetime as

$$a = \frac{c^2}{\varrho} \ . \tag{6.11}$$

With this background, we shall now reduce the metric of the quantum corrected Schwarzschild black hole in the Rindler form to obtain the uniform acceleration of a freely falling particle close to the horizon. The horizons for the quantum corrected Schwarzschild black hole can be determined by setting $f(r) = 0$ in eq.(6.2). The forms of the outer and inner horizons read

$$r_{\pm} = G_0 M \pm \sqrt{G_0^2 M^2 - \tilde{\omega} G_0} \ . \tag{6.12}$$

In eq.(6.12), r_+ and r_- denotes the outer and inner horizons of the black hole.
The outer horizon and $f(r)$ upto order $\tilde{\omega}$ (considering $\tilde{\omega}$ to be small) in natural units are

given by

$$r_+ \approx 2G_0M - \frac{\tilde{\omega}}{2M} \tag{6.13}$$

$$f(r) \approx 1 - \frac{2G_0M}{r} + \frac{2\tilde{\omega}G_0^2M}{r^3} \ . \tag{6.14}$$

We shall now carry out a near horizon expansion with respect to the outer horizon to obtain the Rindler form. The near horizon expansion can be obtained by making a Taylor series expansion of $f(r)$ about the outer horizon r_+. Keeping terms upto first order in the near horizon expansion parameter $(r - r_+)$, we get

$$f(r) \cong f(r_+) + (r - r_+)\frac{df(r)}{dr}\bigg|_{r=r_+} = (r - r_+)f'(r_+) \tag{6.15}$$

where we have used $f(r_+) = 0$. Using this expansion, we obtain the line element in $1 + 1$-dimensions to be

$$\begin{aligned} ds^2 &= f(r)c^2dt^2 - f(r)^{-1}dr^2 \\ &\cong (r - r_+)f'(r_+)c^2dt^2 - \frac{1}{(r - r_+)f'(r_+)}dr^2 \ . \end{aligned} \tag{6.16}$$

We now define a transformation of coordinates to obtain the Rindler form as follows

$$\varrho = 2\sqrt{\frac{r - r_+}{f'(r_+)}} \ . \tag{6.17}$$

Using the above transformation in eq.(6.16), we obtain the Rindler form of the metric in $1 + 1$-dimensions as follows

$$ds^2 \cong \frac{\varrho^2 f'^2(r_+)}{4}c^2dt^2 - d\varrho^2 \ . \tag{6.18}$$

In section 5.2, this part for generic metric is specified with more detail. Here $f'(r_+)$ upto order $\mathcal{O}(\tilde{\omega})$ reads

$$f'(r_+) = \frac{c^2}{2G_0M} - \frac{\hbar\tilde{\omega}c^3}{8G_0^2M^3} \ . \tag{6.19}$$

Comparing eq.(6.9) with eq.(6.18), we get the uniform acceleration corresponding to curves of constant ϱ upto linear order in $\tilde{\omega}$ as

$$\begin{aligned} a = \frac{c^2}{\varrho} &= \frac{c^2}{2\sqrt{\frac{r-r_+}{f'(r_+)}}} \\ &\cong \frac{c^2\sqrt{f'(r_+)}}{2\sqrt{r}}\left(1+\frac{r_+}{2r}\right) \\ &\cong \frac{c^3}{\sqrt{2G_0Mr}}\left[1+\frac{G_0M}{rc^2}-\frac{\hbar\tilde{\omega}c}{8G_0M^2}\left(1+\frac{3G_0M}{c^2r}\right)\right] \ . \end{aligned} \tag{6.20}$$

We shall use this result in the subsequent section to check the validity of the Einstein's equivalence principle.

6.3 Atom falling into the quantum corrected black hole

In this section, we shall calculate the probability of transition from the ground state of a two-level atom falling into a quantum corrected Schwarzschild black hole to the excited state along with the emission of a photon. We will denote the ground state and the excited state by g and e. Before proceeding further, we note that in the quantum corrected black hole geometry the atom trajectory is given by the following equations

$$\begin{aligned} \tau(r) &= -\int \frac{dr}{\sqrt{1-f(r)}} \\ &\approx -\frac{1}{\sqrt{2G_0M}}\left(\frac{2}{3}r^{\frac{3}{2}}-\tilde{\omega}G_0r^{-\frac{1}{2}}\right)+C' \end{aligned} \tag{6.21}$$

$$
\begin{aligned}
t(r) &= -\int \frac{dr}{f(r)\sqrt{1-f(r)}} \\
&\approx -4G_0M\left(\frac{r^{\frac{1}{2}}}{\sqrt{2G_0M}} + \frac{r^{\frac{3}{2}}}{3(2G_0M)^{\frac{3}{2}}} + \frac{1}{2}\ln\frac{\frac{\sqrt{r}}{\sqrt{2G_0M}}-1}{\frac{\sqrt{r}}{\sqrt{2G_0M}}+1}\right) \\
&+\tilde{\omega}G_0\left(\frac{r^{\frac{1}{2}}/2G_0M}{(r-2G_0M)} + \frac{1}{2(2G_0M)^{\frac{3}{2}}}\ln\frac{\frac{\sqrt{r}}{\sqrt{2G_0M}}-1}{\frac{\sqrt{r}}{\sqrt{2G_0M}}+1}\right) + C \qquad (6.22)
\end{aligned}
$$

where C' and C are integration constants. For simplicity we shall set $2G_0M = 1$ during the calculation of the excitation probability of the atom with simultaneous photon emission. For a massless scalar photon with wave function Ψ, the covariant Klein Gordon equation is given as

$$
\frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}g^{\alpha\beta}\partial_\beta\right)\Psi = 0 \ . \qquad (6.23)
$$

In the $s-$wave approximation eq.(6.23) takes the form

$$
\frac{1}{T(t)}\frac{d^2T(t)}{dt^2} - \frac{f(r)}{r^2R(r)}\frac{d}{dr}\left(r^2 f(r)\frac{dR(r)}{dr}\right) = 0 \qquad (6.24)
$$

where the wave function $\Psi(t, r)$ of the scalar photon is written as

$$
\Psi(t, r) = T(t)R(r) \ . \qquad (6.25)
$$

The general solution of eq.(6.24) reads

$$
\Psi_\nu(t, r) = \exp\left[-i\nu t + i\nu\int\frac{dr}{f(r)}\right] \ . \qquad (6.26)
$$

In eq.(6.26), 'ν' describes the photon frequency which is observed by an observer sitting at infinity from the quantum corrected black hole. In this work, we follow the model in [71] consisting of a mirror surrounding the black hole to shield Hawking radiation coming out of it. This set up is identical to that of a Boulware vacuum [197]. Now defining an operator $\hat{\zeta} = |g\rangle\langle e|$, the atom field interaction Hamiltonian can be written as

$$
\hat{H}_I(\tau) = \hbar\mathcal{G}[\hat{b}_\nu\Psi_\nu(t(\tau), r(\tau)) + H.c.][\hat{\zeta}e^{-i\varpi\tau} + H.c.] \ . \qquad (6.27)
$$

In eq.(6.27), $\mathcal{G}$ is the atom-field coupling constant, $\hat{b}_\nu$ is the annihilation operator of the photon and ϖ is the atomic frequency. With the simultaneous emission of a scalar photon, the probability of excitation of the atom is given by

$$
\begin{aligned}
P_{g,0\to e,1} &= \frac{1}{\hbar^2}\left|\int d\tau\langle 1_\nu, e|\hat{H}_I(\tau)|0_\nu, g\rangle\right|^2 \\
&= \mathcal{G}^2\left|\int d\tau e^{i\nu t(\tau)-i\nu r_*(\tau)}e^{i\varpi\tau}\right|^2 \\
&= \mathcal{G}^2\left|\int_\infty^{1-\frac{\tilde{\omega}}{2M}} dr\left(\frac{d\tau}{dr}\right)e^{i\nu t(r)-i\nu r_*(r)}e^{i\varpi\tau(r)}\right|^2 \\
&\approx \mathcal{G}^2\left|\int_1^\infty dr\sqrt{r}\left(1+\frac{\tilde{\omega}}{4Mr^2}\right)\Psi_\nu^*(r)e^{i\varpi\tau(r)}\right|^2
\end{aligned}
\tag{6.28}
$$

where the forms of $\tau(r)$ and $t(r)$ are given in eq.(s)(6.21,6.22). We will now make a change of variables as follows

$$
y = \frac{2\varpi}{3}(r^{\frac{3}{2}}-1) \tag{6.29}
$$

where $\varpi \gg 1$. Using eq.(6.29) in eq.(6.28), the form of $P_{g,0\to e,1}$ is obtained as

$$
P_{g,0\to e,1} = \frac{\mathcal{G}^2}{\varpi^2}\left|\int_0^\infty dy\mathcal{A}_0 e^{-i\nu\mathcal{A}_1(y)}e^{-i\varpi\mathcal{A}_2(y)}\right|^2 \tag{6.30}
$$

where

$$
\mathcal{A}_0 = \left(1+\frac{\tilde{\omega}}{4M}\left(1+\frac{3y}{2\varpi}\right)^{-\frac{4}{3}}\right) \tag{6.31}
$$

$$
\begin{aligned}
\mathcal{A}_1(y) &= 2\left(1+\frac{3y}{2\varpi}\right)^{\frac{1}{3}}+\frac{2}{3}\left(1+\frac{3y}{2\varpi}\right)+\left(1+\frac{3y}{2\varpi}\right)^{\frac{2}{3}} \\
&+2\ln\left[\left(1+\frac{3y}{2\varpi}\right)^{\frac{1}{3}}-1\right]+\frac{\tilde{\omega}}{2M}\Bigg\{-\ln\left[1+\frac{3y}{2\varpi}\right]^{\frac{2}{3}} \\
&+2\ln\left[\left(1+\frac{3y}{2\varpi}\right)^{\frac{1}{3}}-1\right]+\frac{1}{\left(1+\frac{3y}{2\varpi}\right)^{\frac{1}{3}}-1}\Bigg\}
\end{aligned}
\tag{6.32}
$$

$$
\mathcal{A}_2(y) = \frac{2}{3}\left(1+\frac{3y}{2\varpi}\right)-\frac{\tilde{\omega}}{2M}\left(1+\frac{3y}{2\varpi}\right)^{-\frac{1}{3}}. \tag{6.33}
$$

Simplify eq.(6.30), $P_{g,0\to e,1}$ upto linear order in $\tilde{\omega}$ reads

$$P_{g,0\to e,1} \approx \frac{\mathcal{G}^2}{\varpi^2}\left|\int_0^\infty dy\left(1+\frac{\tilde{\omega}}{4M}\right)y^{-\left(2i\nu+\frac{i\tilde{\omega}\nu}{M}\right)}\right. \left. e^{-iz\left(1+\frac{3\nu}{\varpi}+\frac{\tilde{\omega}}{4M}\left(1-\frac{2\nu}{\varpi}\right)\right)}\left(1-\frac{i\tilde{\omega}\nu\varpi}{yM}\right)\right|^2 . \tag{6.34}$$

We now make another change of coordinates as follows

$$z = y\left(1+\frac{3\nu}{\varpi}+\frac{\tilde{\omega}}{4M}\left(1-\frac{2\nu}{\varpi}\right)\right) . \tag{6.35}$$

Using the above coordinate transformation in eq.(6.34), we get

$$\begin{aligned} P_{g,0\to e,1} &\approx \frac{\mathcal{G}^2\left(1+\frac{\tilde{\omega}}{2M}\frac{5\nu}{\varpi+3\nu}\right)}{\varpi^2\left(1+\frac{3\nu}{\varpi}\right)^2}\left|\int_0^\infty dz\ e^{-iz}z^{-\left(2i\nu+\frac{i\tilde{\omega}\nu}{M}\right)}\right. \\ &\quad \left.\left(1-\frac{i\tilde{\omega}\nu\varpi}{zm}\left(1+\frac{3\nu}{\varpi}\right)\right)\right|^2 \\ &\approx \frac{4\pi\mathcal{G}^2\nu\left(1+\frac{\tilde{\omega}}{2M}\frac{\varpi+8\nu}{\varpi+3\nu}\right)}{\varpi^2\left(1+\frac{3\nu}{\varpi}\right)^2}\frac{1}{e^{4\pi\nu\left(1+\frac{\tilde{\omega}}{2M}\right)}-1} . \end{aligned} \tag{6.36}$$

The absorption probability on the other hand is given by

$$P_{e,0\to g,1} \approx \frac{4\pi\mathcal{G}^2\nu\left(1+\frac{\tilde{\omega}}{2M}\frac{\varpi-8\nu}{\varpi-3\nu}\right)}{\varpi^2\left(1-\frac{3\nu}{\varpi}\right)^2}\frac{1}{1-e^{-4\pi\nu\left(1+\frac{\tilde{\omega}}{2M}\right)}} . \tag{6.37}$$

It can be observed that for a very high photon frequency, the excitation probability becomes considerably smaller. The atomic frequency is considerably higher and in our case we consider $\varpi \gg \nu$. With dimensional reconstruction and considering the case when atomic frequency is substantially higher than the emitted photon frequency, we obtain the excitation probability to be

$$P_{g,0\to e,1} \approx \frac{4\pi\mathcal{G}^2\nu\left(\frac{2G_0M}{c^3}+\frac{\hbar\tilde{\omega}}{2Mc^2}\right)}{\varpi^2}\frac{1}{e^{4\pi\nu\left(\frac{2G_0M}{c^3}+\frac{\hbar\tilde{\omega}}{2Mc^2}\right)}-1} . \tag{6.38}$$

Now for the mirror-atom system [71, 193], the excitation probability for the case when the mirror is accelerating with respect to the fixed atom, reads

$$P^{AM}_{g,0\to e,1} = \frac{4\pi\mathcal{G}^2\nu c}{a\varpi^2}\frac{1}{e^{\frac{2\pi c\nu}{a}}-1} \tag{6.39}$$

where ν is the local photon frequency in the reference frame of mirror at Rindler space. It is observed in [71] that the Einstein principle of equivalence holds for the case of a Schwarzschild black hole. In this work, we shall check its status for the quantum corrected Schwarzschild black hole. At first we replace ν by ν_∞ (photon frequency observed by a distant observer) in eq.(6.38) to differentiate it from the local photon frequency at Rindler space in (6.39). Now the relation between local frequency ν and ν_∞ in quantum corrected Schwarzschild black hole space is connected by the gravitational red-shift factor as

$$\nu = \frac{\nu_\infty}{\sqrt{f(r)}} \implies \nu_\infty \cong \nu\sqrt{(r-r_+)f'(r_+)} \tag{6.40}$$

where we have substituted $f(r)$ from eq.(6.15) to obtain the analytical form of ν_∞. From the first line of eq.(6.20) we can write

$$\begin{aligned}\sqrt{r-r_+} &= \frac{c^2}{2a}\sqrt{f'(r_+)}\\ \implies \nu\sqrt{(r-r_+)f'(r_+)} &= \nu_\infty = \nu f'(r_+)\frac{c^2}{2a}\\ \implies \nu_\infty &= \frac{\nu c^2}{2a}\left[\frac{c^2}{2G_0M}-\frac{\hbar\tilde{\omega}c^3}{8G_0^2M^3}\right]\end{aligned} \tag{6.41}$$

where we have used eq.(6.19) to replace $f'(r_+)$ in the last line of the above equation with proper dimensional reconstruction. For a distant observer the excitation probability in eq.(6.38) takes the following form

$$\mathcal{P}\Big|_{\nu=\nu_\infty} \cong \frac{4\pi\mathcal{G}^2\nu_\infty\left(\frac{2G_0M}{c^3}+\frac{\hbar\tilde{\omega}}{2Mc^2}\right)}{\varpi^2}\frac{1}{e^{4\pi\nu_\infty\left(\frac{2G_0M}{c^3}+\frac{\hbar\tilde{\omega}}{2Mc^2}\right)}-1} \tag{6.42}$$

where

$$\begin{aligned}\nu_\infty\left(\frac{2G_0M}{c^3}+\frac{\hbar\tilde{\omega}}{2Mc^2}\right)=&\frac{\nu c^2}{2a}\left[\frac{c^2}{2G_0M}-\frac{\hbar\tilde{\omega}c^3}{8G_0^2M^3}\right]\\&\times\left[\frac{2G_0M}{c^3}+\frac{\hbar\tilde{\omega}}{2Mc^2}\right]\\=&\frac{\nu c}{2a}\left(1+\mathcal{O}(\tilde{\omega}^2)\right)\cong\frac{\nu c}{2a}.\end{aligned} \tag{6.43}$$

Using relation (6.43) in eq.(6.42), we obtain the probability as follows

$$\mathcal{P}=\frac{2\pi\mathcal{G}^2\nu c}{a\varpi^2}\frac{1}{e^{\frac{2\pi c\nu}{a}}-1}\ . \tag{6.44}$$

The excitation probability in eq.(6.44) is identical to eq.(6.39) upto a constant factor. This particular insight into the equivalence principle has a very subtle difference than the usual description of the equivalence principle using the elevator description. In the case of the accelerating mirror, the normal modes of the fields get modified and for the freely falling atom the gravitational field of the black hole is responsible for the same effect. What we observe interestingly is that the emitted radiation is identical for the two cases. This implies that the effect of the mirror acceleration and that of gravitational field is same on the atom. This result indicates that the aforementioned description of the equivalence principle holds exactly in case of this renormalization group improved Schwarzschild black hole as well.

The absorption probability in eq.(6.37) for $\varpi \gg \nu$ reads

$$P_{e,0\to g,1}\approx\frac{4\pi\mathcal{G}^2\nu\left(\frac{2G_0M}{c^3}+\frac{\hbar\tilde{\omega}}{2Mc^2}\right)}{\varpi^2}\frac{1}{1-e^{-4\pi\nu\left(\frac{2G_0M}{c^3}+\frac{\hbar\tilde{\omega}}{2Mc^2}\right)}}\ . \tag{6.45}$$

In the next section, we shall use this result to compute the HBAR entropy.

6.4 Atom falling into a generic black hole spacetime

The previous sections dealt with the case of a quantum corrected black hole. We will now consider a generic black hole geometry with lapse function $f(r)$ having an event horizon at r_+. The excitation probability is given as follows

$$\begin{aligned}\mathcal{P}_{f,r_+} =&\mathcal{G}^2\left|\int_{\infty}^{r_+} dr\frac{d\tau}{dr}e^{i\nu t(r)-i\nu r_*(r)}e^{i\varpi\tau(r)}\right|^2\\ =&\mathcal{G}^2\left|\int_{r_+}^{\infty} dr\frac{1}{\sqrt{1-f(r)}}e^{-i\nu\int\frac{dr}{f(r)\sqrt{1-f(r)}}-i\nu\int\frac{dr}{f(r)}}\right.\\ &\left.\times\, e^{-i\varpi\int\frac{dr}{\sqrt{1-f(r)}}}\right|^2 .\end{aligned} \tag{6.46}$$

We have used the form of $\tau(r)$, $t(r)$ and $r_*(r)$ to obtain the form in the above line of eq.(6.46).Using the near horizon expansion of $f(r)$ from eq.(6.15) in the above equation, we obtain the following expression for the excitation probability

$$\begin{aligned}\mathcal{P}_{f,r_+} =&\mathcal{G}^2\left|\int_{r_+}^{\infty} dr\frac{e^{-i\nu\int\frac{dr}{(r-r_+)f'(r_+)\sqrt{1-(r-r_+)f'(r_+)}}}}{\sqrt{1-(r-r_+)f'(r_+)}}\right.\\ &\left.\times\, e^{-i\nu\int\frac{dr}{(r-r_+)f'(r_+)}}e^{-i\varpi\int\frac{dr}{\sqrt{1-(r-r_+)f'(r_+)}}}\right|^2\\ =&\mathcal{G}^2\left|\int_{r_+}^{\infty} dr\frac{e^{-\frac{2i\nu}{f'(r_+)}\ln\left(1-\sqrt{1-(r-r_+)f'(r_+)}\right)}}{\sqrt{1-(r-r_+)f'(r_+)}}\right.\\ &\left.\times\, e^{\frac{2i\varpi}{f'(r_+)}\sqrt{1-(r-r_+)f'(r_+)}}\right|^2 .\end{aligned} \tag{6.47}$$

We will now make a change of variables as follows

$$r-r_+=\frac{\kappa}{\varpi} \tag{6.48}$$

where $f'(r_+),\nu,\kappa\ll\varpi$ (in natural units).

Using the above change in variables from eq.(6.48) in eq.(6.47), we obtain the following form of the excitation probability

$$\begin{aligned}\mathcal{P}_{f,r_+} \cong & \frac{\mathcal{G}^2}{\varpi^2}\bigg|\int_0^\infty d\kappa\left(1+\frac{\kappa}{2\varpi}f'(r_+)\right)e^{-\frac{2i\nu}{f'(r_+)}\ln\left(\frac{\kappa}{2\varpi}f'(r_+)\right)} \\ & \times e^{\frac{2i\varpi}{f'(r_+)}\left(1-\frac{\kappa}{2\varpi}f'(r_+)\right)}\bigg|^2 \\ = & \frac{\mathcal{G}^2}{\varpi^2}\bigg|\int_0^\infty d\kappa\left(1+\frac{\kappa}{2\varpi}f'(r_+)\right)\kappa^{-\frac{2i\nu}{f'(r_+)}}e^{-i\kappa}\bigg|^2 \\ = & \frac{4\pi\mathcal{G}^2\nu}{f'(r_+)\varpi^2}\left(\left(1-\frac{\nu}{\varpi}\right)^2+\frac{f'^2(r_+)}{4\varpi^2}\right)\frac{1}{e^{\frac{4\pi\nu}{f'(r_+)}}-1}\ .\end{aligned} \tag{6.49}$$

As $\nu, f'(r_+) \ll \varpi$ (in natural units), the probability can be recast in the following form

$$\mathcal{P}_{f,r_+} \cong \frac{4\pi\mathcal{G}^2\nu}{f'(r_+)\varpi^2}\frac{1}{e^{\frac{4\pi\nu}{f'(r_+)}}-1}\ . \tag{6.50}$$

We can again recast the probability in terms of ν_∞ as follows (as was done earlier in eq.(6.42))

$$\begin{aligned}\mathcal{P}_{f,r_+} = & \frac{4\pi\mathcal{G}^2\nu_\infty}{f'(r_+)\varpi^2}\frac{1}{e^{\frac{4\pi\nu_\infty}{f'(r_+)}}-1} \\ = & \frac{4\pi\mathcal{G}^2\nu\sqrt{(r-r_+)f'(r_+)}}{f'(r_+)\varpi^2}\frac{1}{e^{\frac{4\pi\nu\sqrt{(r-r_+)f'(r_+)}}{f'(r_+)}}-1}\ .\end{aligned} \tag{6.51}$$

Using eq.(6.41) in eq.(6.51), we get the following form of the probability

$$\begin{aligned}\mathcal{P}_{f,r_+} = & \frac{4\pi\mathcal{G}^2\left(\frac{\nu}{2a}f'(r_+)\right)}{f'(r_+)\varpi^2}\frac{1}{e^{\frac{4\pi\left(\frac{\nu}{2a}f'(r_+)\right)}{f'(r_+)}}-1} \\ = & \frac{2\pi\mathcal{G}^2\nu}{a\varpi^2}\frac{1}{e^{\frac{2\pi\nu}{a}}-1}\ .\end{aligned} \tag{6.52}$$

With proper dimensional reconstruction, eq.(6.52) can be rewritten as follows

$$\mathcal{P}_{f,r_+} = \frac{2\pi\mathcal{G}^2\nu c}{a\varpi^2}\frac{1}{e^{\frac{2\pi\nu c}{a}}-1}\ . \tag{6.53}$$

Hence, following the same arguments after eq.(6.44), we can conclude that the equivalence principle holds for a generic black hole spacetime.

6.5 Modified HBAR entropy

The term *horizon brightened acceleration radiation entropy* (HBAR entropy) was first introduced in [71]. In this section we will calculate the HBAR entropy for the quantum corrected black hole background. We are considering a case when two-level atoms with transition frequency ϖ are falling into the event horizon of the black hole at a rate κ. In this work we have used the quantum statistical approach to determine the entropy and hence we have used the density matrix formalism. If the microscopic change in the field density matrix is $\delta\rho_a$ for one atom then the overall macroscopic change in the field density matrix for $\Delta\mathcal{N}$ atoms is given by [71]

$$\Delta\rho = \sum_a \delta\rho_a = \Delta\mathcal{N}\delta\rho \tag{6.54}$$

where

$$\frac{\Delta\mathcal{N}}{\Delta t} = \kappa \ . \tag{6.55}$$

Putting $\Delta\mathcal{N}$ in eq.(6.55), we obtain

$$\frac{\Delta\rho}{\Delta t} = \kappa\delta\rho \ . \tag{6.56}$$

We now use the Lindblad master equation for the density matrix given by

$$\begin{aligned} \frac{d\rho}{dt} =& -\frac{\Gamma_{abs}}{2}\left(\rho b^\dagger b + b^\dagger b\rho - 2b\rho b^\dagger\right) \\ & -\frac{\Gamma_{exc}}{2}\left(\rho b b^\dagger + b b^\dagger\rho - 2b^\dagger\rho b\right) \end{aligned} \tag{6.57}$$

where Γ_{exc} is the excitation rate and Γ_{abs} is the absorption rate given by $\Gamma_{exc/abs} = \kappa P_{exc/abs}$ with $P_{exc/abs}$ being given by eq.(s)(6.36,6.37). Taking the expectation of eq.(6.57) with respect to some arbitrary state $|n\rangle$, we obtain

$$\begin{aligned}\dot{\rho}_{n,n} =& -\Gamma_{abs}\left(n\rho_{n,n} - (n+1)\rho_{n+1,n+1}\right) \\ & -\Gamma_{exc}\left((n+1)\rho_{n,n} - n\rho_{n-1,n-1}\right) \ .\end{aligned} \tag{6.58}$$

The steady state solution is now used to obtain the HBAR entropy. So we set $\dot{\rho}_{n,n} = 0$ in eq.(6.58) and for $n = 0$ we obtain the relation between $\rho_{1,1}$ and $\rho_{0,0}$ as follows

$$\rho_{1,1} = \frac{\Gamma_{exc}}{\Gamma_{abs}}\rho_{0,0} \ . \tag{6.59}$$

Repeating this procedure, we finally obtain

$$\rho_{n,n} = \left(\frac{\Gamma_{exc}}{\Gamma_{abs}}\right)^n \rho_{0,0} \ . \tag{6.60}$$

To obtain $\rho_{0,0}$ in the above relation, we use the condition $\mathrm{Tr}(\rho) = 1$. This gives

$$\begin{aligned}&\sum_n \rho_{n,n} = 1 \implies \rho_{0,0}\sum_n \left(\frac{\Gamma_{exc}}{\Gamma_{abs}}\right)^n = 1 \\ \implies & \rho_{0,0} = 1 - \frac{\Gamma_{exc}}{\Gamma_{abs}} \ .\end{aligned} \tag{6.61}$$

Using $\rho_{0,0}$ from the above equation in eq.(6.60), we obtain the steady state solution of the density matrix to be

$$\rho^{\mathcal{S}}_{n,n} = \left(\frac{\Gamma_{exc}}{\Gamma_{abs}}\right)^n \left(1 - \frac{\Gamma_{exc}}{\Gamma_{abs}}\right) \tag{6.62}$$

where $\frac{\Gamma_{exc}}{\Gamma_{abs}}$ is given by (in the approximation $\varpi \gg \nu$)

$$\frac{\Gamma_{exc}}{\Gamma_{abs}} \approx \left(1 + \frac{\hbar\tilde{\omega}}{\mathcal{R}Mc}\frac{5\nu}{\varpi}\right) e^{-4\pi\nu\left(\frac{\mathcal{R}}{c} + \frac{\hbar\tilde{\omega}}{2Mc^2}\right)} \ . \tag{6.63}$$

Here, $\mathcal{R} = \frac{2GM}{c^2}$. The von-Neumann entropy for the system is given by

$$S_\rho = -k_B \sum_{n,\nu} \rho_{n,n} \ln(\rho_{n,n}) \tag{6.64}$$

and the rate of change of entropy due to the generation of real photons is obtained as

$$\dot{S}_\rho = -k_B \sum_{n,\nu} \dot{\rho}_{n,n} \ln(\rho_{n,n}) \ . \tag{6.65}$$

The rate of change of the entropy, using the steady state density matrix solution, is given as

$$\dot{S}_\rho \approx -k_B \sum_{n,\nu} \dot{\rho}_{n,n} \ln(\rho^{\mathcal{S}}_{n,n}) \ . \tag{6.66}$$

Using the form of the density matrix $\rho^{\mathcal{S}}_{n,n}$ in eq.(6.62), we obtain

$$\begin{aligned} \dot{S}_\rho =& 4\pi k_B \left(\frac{\mathcal{R}}{c} + \frac{\hbar\tilde{\omega}}{2Mc^2} \right) \sum_\nu \left(\sum_n n \dot{\rho}_{n,n} \right) \nu \\ & - k_B \sum_\nu \left(\sum_n n\dot{\rho}_{n,n} \right) \ln \left[1 + \frac{\hbar\tilde{\omega}}{\mathcal{R}Mc} \frac{5\nu}{\varpi} \right] \\ \approx & 4\pi k_B \left(\frac{\mathcal{R}}{c} + \frac{\hbar\tilde{\omega}}{2Mc^2} \right) \sum_\nu \dot{\bar{n}}_\nu \nu - \frac{5\hbar\tilde{\omega}k_B}{\mathcal{R}\varpi Mc} \sum_\nu \dot{\bar{n}}_\nu \nu \ . \end{aligned} \tag{6.67}$$

In eq.(6.67), $\dot{\bar{n}}_\nu$ is the flux due to photons from the atoms freely falling in the black hole and the total rate of energy loss due to emitted photons is $\hbar \sum_\nu \dot{\bar{n}}_\nu \nu = \dot{m}_p c^2$. The black hole area in our case is given by

$$\begin{aligned} A_{Qbh} &= 4\pi r_+^2 \\ &\approx \frac{16\pi G_0^2 M^2}{c^4} - \frac{8\pi\hbar\tilde{\omega}G_0}{c^3} \ . \end{aligned} \tag{6.68}$$

Taking time derivative of the both sides in eq.(6.68), we get

$$\dot{A}_{Qbh} = \frac{32\pi G_0^2 M \dot{M}}{c^4} \tag{6.69}$$

where $\dot{M} = \dot{m}_p + \dot{m}_{atom}$. We now define the rate of change in the black hole area due to emitting photons as follows

$$\dot{A}_p = \frac{32\pi G_0^2 M \dot{m}_p}{c^4} \ . \tag{6.70}$$

From these results we can infer that when no atoms are falling in the black hole, then A_p should be equivalent to the area of the black hole and A_{atom} should be zero. A better way to understand the above result is as follows. Before the atom crosses the black hole event horizon, that is, before the atom contributes to the black hole mass, the freely falling atom emits HBAR radiation. Therefore, one can separate in time the change of the black hole entropy associated with HBAR radiation from an atom and that associated with that atom's mass.
The final form of $\dot{S}_\rho$ in terms of A_p can therefore be written as

$$\begin{aligned}\dot{S}_\rho \approx &\frac{k_B c^3}{4\hbar G_0}\dot{A}_p + \tilde{\omega}\pi k_B \frac{d}{dt}(\ln A_p) + \frac{10\tilde{\omega}\sqrt{\pi}k_B c}{\varpi}\frac{d}{dt}\left(A_p^{-\frac{1}{2}}\right) \\ = &\frac{d}{dt}\left(\frac{k_B c^3}{4\hbar G_0}A_p + \tilde{\omega}\pi k_B \ln A_p + \frac{10\tilde{\omega}\sqrt{\pi}k_B c}{\varpi}A_p^{-\frac{1}{2}}\right) \ .\end{aligned} \tag{6.71}$$

We find that the horizon brightened acceleration radiation entropy for the case of a simple Schwarzschild black hole [71] gets modified with the time derivatives of a log term and an inverse square root term due to quantum gravity corrections introduced in the Schwarzschild black hole. This indicates that logarithmic corrections also appear in HBAR entropy just as it appears in usual Bekenstein-Hawking black hole entropy [4–7, 196].

6.6 Conclusion

In this work we have considered the case of a two-level atom freely falling near the outer horizon of a renormalization group improved Schwarzschild black hole. The entire system is analysed by considering that the outer horizon of the black hole is surrounded with a mirror to prevent infalling atoms from interacting with the Hawking radiation. Further, the mirror ensures that the initial state of the field appears vacuum-like in the reference frame of the external observer. Our investigation in this context involved the computation of the atom-field excitation probability and the calculation of the rate of change of the horizon brightened acceleration radiation entropy or the HBAR entropy.

The excitation probability obtained in this work involves a Planck like factor indicating emission of real photons in the virtual process. Interestingly, we find that even with quantum gravity corrections, the Einstein equivalence principle holds indicating that the principle may be a fundamental reality even in a quantum gravity setting. We then extended this calculation for a generic black hole metric and observed that the equivalence principle holds in a general setting as well. For the next part of the work we have calculated the HBAR entropy using the quantum optics approach [71] and we have observed that the HBAR entropy has a Bekenstein-Hawking like entropy term along with a logarithmic and inverse square root quantum gravity term. The nature of the corrections appearing in the HBAR entropy are remarkably the same as those obtained in case of usual Bekenstein-Hawking entropy of a black hole.

Chapter 7

Conclusions

The quest for a theory of quantum gravity was hampered by the non-renomalizable nature of gravity, but in recent years, the use of functional renormalization group (FRG) techniques has permitted a thorough exploration of gravity. As discussed, the asymptotic safety scenario for quantum gravity is a viable conjecture to build a consistent and predictive quantum theory for the gravitational interaction since the non-Gaussian fixed point (NGFP) makes it safe in ultraviolet regime. In this thesis, we have extensively used renormalization group (RG) approaches in cosmology to examine the behaviour of the standard Einstein equations in the late universe and to investigate the thermodynamics of black holes.

Effective Average Action (EAA), which is the fundamental idea underlying modern Functional Renormalization Group (FRG) approaches, was first described in chapter 2 for the basic scalar field theory. Our attention next turned to the gravitational renormalization group flow projected onto the Einstein-Hilbert subspace, from which the beta functions for the cosmological constant and Newton's coupling are deduced. In order to find the next-to-leading order solution of the cosmological constant Λ and the Newton's constant G, the flow equations at the infrared regime are finally solved.

In this formalism of the exact renormalization group flow of the effective average action for gravity, we investigated cosmology for the anisotropic Bianchi type-I and FLRW universe at late times while accounting for quantum gravitational corrections. Firstly, having included the higher powers of 1/t in the expression for the infrared cutoff scale k, the anisotropic Bianchi-I cosmology is carried out extensively in the consistency equation approach. This approach calls for the covariant conservation of the energy momentum tensor along with the covariant conservation of the right hand side of Einstein equation, which in turn supplied with an ordinary continuity equation and a consistency condition incorporating quantum corrections. Then, we examined how the flow of anisotropic Bianchi-I cosmology is impacted by quantum gravitational effects for known matter such as dust, radiation, and stiff matter by improving the Einstein solution with the quantum corrected G and Λ. The consistency condition resulting from Einstein equations show that the Bianchi-I anisotropic cosmic universe gradually evolves into a FLRW universe at later times for $0 < \Omega < 1$ if a perfect fluid with an equation of state $p = \Omega\rho$ is present. Radiation is one example of this. We discover that the FLRW isotropic universe does not necessarily flow from the Bianchi-I universe for dust ($\Omega = 0$). We determine a bound on the cut off parameter ξ for stiff matter ($\Omega = 1$) that guarantees Bianchi-I universe flows to the isotropic FLRW universe at late times if ξ^2 equals its maximum value, but not otherwise. Additionally, we discover that there is a chance of obtaining a Kasner-like solution in this scenario.

We then discuss the impact of this scale dependency on the FLRW universe accounting quantum gravitational correction as before, specifically how the scale factor behaves over time when utilising various infrared cut-off scales in the context of a modified continuity equation approach. The modified continuity approach differs from the prior one in that, unlike the consistency equation approach, it only requires the covariant conservation of the right hand side of the Einstein equation. For the cut off choice $k = \frac{\xi}{t}$, we note that an $\mathcal{O}(1/t)$ term that was absent from the consistency equation approach, arises in the quantum correction of the scale factor. We also estimated the rate at which entropy is produced for this particular cut-off and found that it receives a time-dependent quantum correction at $\mathcal{O}(1/t^2)$ for radiation assuming very small non-adiabiticity. We looked at two other cut-off scale options, namely, $k = \varepsilon H^{\frac{3}{4}}$, and $k = \gamma H$ since it's possible that they have a simpler functional relationship with the Hubble parameter than with cosmological time. One of the primary discoveries in that chapter is the time evolution of the scale factors with these cut-off choices.

We then move to investigate the thermodynamics and phase transition of the renormalized group improved asymptotically safe Schwarzchild black hole taking the quantum corrections by improving solutions of the classical Einstein equation. Identifying $G(r) \equiv G(k = k(r))$, we investigated several thermodynamic quantities with the generic parameter γ for the quantum corrected Schwarzschild metric, such as the Hawking temperature, specific heat, and entropy. We have noticed that the coefficient of the leading quantum correction, namely, the logarithmic correction gets affected by the presence of γ. Once the black hole has decreased its mass to $M = M_{cr}$, we observe how the vanishing temperature of the critical black hole causes the evaporation process to end. This provides support to the notion that a cold remnant represents the evaporation's final state. By calculating the on-shell free energy, we discovered that the quantum corrected Schwarzschild black hole maintained in a concentric spherical cavity never undergoes a Hawking-Page phase transition because the black hole state always prevails for all temperatures. It stands in sharp contrast to the typical Schwarzschild black hole scenario contained in a cavity, where the Hawking-Page phase transition occurs. This observation suggests that the first order phase transition is avoided by the quantum corrected black hole.

Finally, we looked at a two-level atom that was falling freely close to the outer horizon of quantum improved Schwarzschild black hole. Interestingly, the Einstein equivalence principle is found to stay true even with quantum gravity corrections, suggesting that the principle may be a fundamental fact even in a quantum gravity scenario. We have also studied the horizon brightening acceleration radiation entropy (HBAR entropy) and found that it contains a Bekenstein-Hawking-like entropy term in addition to a logarithmic and inverse-square-root quantum gravity term. The HBAR entropy corrections are strikingly similar in kind to those found for the standard Bekenstein-Hawking entropy of a black hole.

References

[1] Clifford M. Will. The confrontation between general relativity and experiment. *Living Reviews in Relativity*, 17(1), June 2014.

[2] Clifford M. Will. *Theory and Experiment in Gravitational Physics.* Cambridge University Press, September 2018.

[3] S W Hawking and R Penrose. The singularities of gravitational collapse and cosmology. *Proc. R. Soc. Lond.*, 314(1519):529–548, January 1970.

[4] J D Bekenstein. Black holes and the second law. *Lett. al nuovo cimento*, 4(15):737–740, August 1972.

[5] Jacob D Bekenstein. Black holes and entropy. *Phys. Rev. D Part. Fields*, 7(8):2333–2346, April 1973.

[6] S W Hawking. Black hole explosions? *Nature*, 248(5443):30–31, March 1974.

[7] S. W. Hawking. Particle creation by black holes. *Communications In Mathematical Physics*, 43(3):199–220, August 1975.

[8] Ryoyu Utiyama. Invariant theoretical interpretation of interaction. *Physical Review*, 101(5):1597–1607, March 1956.

[9] Steven Weinberg. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* John Wiley and Sons, New York, 1972.

[10] F. W. Hehl, G. D. Kerlick, and P. Von Der Heyde. On a New Metric Affine Theory of Gravitation. *Phys. Lett. B*, 63:446–448, 1976.

[11] Friedrich W. Hehl, Paul von der Heyde, G. David Kerlick, and James M. Nester. General relativity with spin and torsion: Foundations and prospects. *Reviews of Modern Physics*, 48(3):393–416, jul 1976.

[12] M. Blagojević, F. W. Hehl, and T. W. B. Kibble. *Gauge Theories of Gravitation.* IMPERIAL COLLEGE PRESS, November 2011.

[13] Carlo Rovelli. *Quantum gravity.* Cambridge university press, 2004.

[14] Thomas Thiemann. Lectures on loop quantum gravity. In *Quantum Gravity*, Lecture notes in physics, pages 41–135. Springer Berlin Heidelberg, Berlin, Heidelberg, 2003.

[15] J. Wess and J. Bagger. *Supersymmetry and supergravity.* Princeton University Press, Princeton, NJ, USA, 1992.

[16] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson, and R. Roiban. Ultraviolet behavior of $\mathcal{N} = 8$ supergravity at four loops. *Phys. Rev. Lett.*, 103:081301, Aug 2009.

[17] Joseph Polchinski. *String Theory*, volume 1 of *Cambridge Monographs on Mathematical Physics.* Cambridge University Press, 1998.

[18] Joseph Polchinski. *String Theory*, volume 2 of *Cambridge Monographs on Mathematical Physics.* Cambridge University Press, 1998.

[19] Elias Kiritsis. *String theory in a nutshell.* Princeton University Press, USA, 2019.

[20] John F. Donoghue. General relativity as an effective field theory: The leading quantum corrections. *Phys. Rev. D*, 50:3874–3888, Sep 1994.

[21] Erik Verlinde. On the origin of gravity and the laws of newton. *J. High Energy Phys.*, 2011(4), April 2011.

[22] T. Padmanabhan. Thermodynamical aspects of gravity: new insights. *Rep. Prog. Phys.*, 73(4):046901, April 2010.

[23] Steven Weinberg. *ULTRAVIOLET DIVERGENCES IN QUANTUM THEORIES OF GRAVITATION*, pages 790–831. 1980.

[24] David J. Gross and Frank Wilczek. Ultraviolet behavior of non-abelian gauge theories. *Phys. Rev. Lett.*, 30:1343–1346, Jun 1973.

[25] H. David Politzer. Reliable perturbative results for strong interactions? *Phys. Rev. Lett.*, 30:1346–1349, Jun 1973.

[26] M. Reuter. Nonperturbative evolution equation for quantum gravity. *Phys. Rev. D*, 57:971–985, January 1998.

[27] O. Lauscher and M. Reuter. Ultraviolet fixed point and generalized flow equation of quantum gravity. *Phys. Rev. D*, 65:025013, December 2001.

[28] O. Lauscher and M. Reuter. Flow equation of quantum Einstein gravity in a higher-derivative truncation. *Phys. Rev. D*, 66:025026, July 2002.

[29] Wataru Souma. Nontrivial ultraviolet fixed point in quantum gravity. *Progress of Theoretical Physics*, 102(1):181–195, July 1999.

[30] Roberto Percacci. Further evidence for a gravitational fixed point. *Physical Review D*, 73(4), feb 2006.

[31] Max Niedermaier and Martin Reuter. The asymptotic safety scenario in quantum gravity. *Living Reviews in Relativity*, 9(1), dec 2006.

[32] M Niedermaier. The asymptotic safety scenario in quantum gravity: an introduction. *Classical and Quantum Gravity*, 24(18):R171–R230, aug 2007.

[33] M. Niedermaier. Gravitational fixed points and asymptotic safety from perturbation theory. *Nuclear Physics B*, 833(3):226–270, jul 2010.

[34] K. Falls, D. Litim, K. Nikolakopoulos, and C. Rahmede. Further evidence for asymptotic safety of quantum gravity. *Physical Review D*, 93(10), may 2016.

[35] Roberto Percacci and Daniele Perini. Constraints on matter from asymptotic safety. *Phys. Rev. D Part. Fields*, 67(8), April 2003.

[36] Roberto Percacci and Daniele Perini. Asymptotic safety of gravity coupled to matter. *Phys. Rev. D Part. Fields*, 68(4), August 2003.

[37] Nicolai Christiansen, Daniel F. Litim, Jan M. Pawlowski, and Manuel Reichert. Asymptotic safety of gravity with matter. *Physical Review D*, 97(10), may 2018.

[38] Astrid Eichhorn, Stefan Lippoldt, and Vedran Skrinjar. Nonminimal hints for asymptotic safety. *Physical Review D*, 97(2), jan 2018.

[39] Astrid Eichhorn and Aaron Held. Viability of quantum-gravity induced ultraviolet completions for matter. *Phys. Rev. D*, 96:086025, Oct 2017.

[40] Yuta Hamada and Masatoshi Yamada. Asymptotic safety of higher derivative quantum gravity non-minimally coupled with a matter system. *Journal of High Energy Physics*, 2017(8), aug 2017.

[41] Daniel F. Litim and Matthew J. Trott. Asymptotic safety of scalar field theories. *Phys. Rev. D*, 98:125006, Dec 2018.

[42] Rituparna Mandal, Sunandan Gangopadhyay, and Amitabha Lahiri. Cosmology of Bianchi type-I metric using renormalization group approach for quantum gravity. *Classical and Quantum Gravity*, 37(6):065012, feb 2020.

[43] Sunandan Gangopadhyay, Rituparna Mandal, and Amitabha Lahiri. Bianchi-I Cosmology in Quantum Gravity. *Springer Proc. Phys.*, 277:925–929, 2022.

[44] Rituparna Mandal, Sunandan Gangopadhyay, and Amitabha Lahiri. Cosmology with modified continuity equation in asymptotically safe gravity. *The European Physical Journal Plus*, 137(10), October 2022.

[45] Rituparna Mandal and Sunandan Gangopadhyay. Black hole thermodynamics in asymptotically safe gravity. *General Relativity and Gravitation*, 54(12):159, dec 2022.

[46] Soham Sen, Rituparna Mandal, and Sunandan Gangopadhyay. Equivalence principle and hbar entropy of an atom falling into a quantum corrected black hole. *Phys. Rev. D*, 105:085007, Apr 2022.

[47] M. Reuter and F. Saueressig. Renormalization group flow of quantum gravity in the Einstein-Hilbert truncation. *Phys. Rev. D*, 65:065016, Feb 2002.

[48] O Lauscher and M Reuter. Flow equation of quantum Einstein gravity in a higher-derivative truncation. *Physical Review D*, 66(2), July 2002.

[49] L. Granda and S. Odintsov. Effective average action and nonperturbative renormalization group equation in higher derivative quantum gravity. 4, 06 1998.

[50] Steven Weinberg. Asymptotically safe inflation. *Phys. Rev. D*, 81:083535, Apr 2010.

[51] A. Bonanno and M. Reuter. Cosmology of the planck era from a renormalization group for quantum gravity. *Phys. Rev. D*, 65:043508, Jan 2002.

[52] Benjamin Koch. Renormalization group and black hole production in large extra dimensions. *Physics Letters B*, 663(4):334–337, 2008.

[53] Javier Grande, Joan Solà Peracaula, S. Basilakos, and Manolis Plionis. Hubble expansion and structure formation in the "running flrw model" of the cosmic evolution. *Journal of Cosmology and Astroparticle Physics - JCAP*, 8, 03 2011.

[54] Yi-Fu Cai and Damien A. Easson. Asymptotically safe gravity as a scalar-tensor theory and its cosmological implications. *Phys. Rev. D*, 84:103502, Nov 2011.

[55] M Reuter and F Saueressig. From big bang to asymptotic de sitter: complete cosmologies in a quantum gravity framework. *Journal of Cosmology and Astroparticle Physics*, 2005(09):012–012, sep 2005.

[56] Alfio Bonanno and Sante Carloni. Dynamical system analysis of cosmologies with running cosmological constant from quantum Einstein gravity. *New Journal of Physics*, 14(2):025008, feb 2012.

[57] A. Bonanno and M. Reuter. Cosmology with self-adjusting vacuum energy density from a renormalization group fixed point. *Physics Letters B*, 527(1):9–17, 2002.

[58] A. BONANNO and M. REUTER. Cosmological perturbations in renormalization group derived cosmologies. *International Journal of Modern Physics D*, 13(01):107–121, jan 2004.

[59] Adriano Contillo. Evolution of cosmological perturbations in a renormalization-group-driven inflationary scenario. *Phys. Rev. D*, 83:085016, Apr 2011.

[60] Alfio Bonanno and Martin Reuter. Entropy production during asymptotically safe inflation. *Entropy*, 13(1):274–292, 2011.

[61] Mark Hindmarsh, Daniel Litim, and Christoph Rahmede. Asymptotically safe cosmology. *Journal of Cosmology and Astroparticle Physics*, 2011(07):019–019, jul 2011.

[62] Benjamin Koch and Israel Ramirez. Exact renormalization group with optimal scale and its application to cosmology. *Classical and Quantum Gravity*, 28(5):055008, feb 2011.

[63] S. W. Hawking. Breakdown of predictability in gravitational collapse. *Phys. Rev. D*, 14:2460–2473, Nov 1976.

[64] W. G. Unruh. Notes on black-hole evaporation. *Phys. Rev. D*, 14:870–892, Aug 1976.

[65] William G. Unruh and Robert M. Wald. What happens when an accelerating observer detects a rindler particle. *Phys. Rev. D*, 29:1047–1056, Mar 1984.

[66] P. C. W. Davies, S. A. Fulling, and W. G. Unruh. Energy-momentum tensor near an evaporating black hole. *Phys. Rev. D*, 13:2720–2723, May 1976.

[67] P C W Davies. Thermodynamics of black holes. *Reports on Progress in Physics*, 41(8):1313–1355, aug 1978.

[68] P. C. W. Davies, S. A. Fulling, and Roger Penrose. Radiation from moving mirrors and from black holes. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 356(1685):237–257, 1977.

[69] Marlan O. Scully, Vitaly V. Kocharovsky, Alexey Belyanin, Edward Fry, and Federico Capasso. Enhancing acceleration radiation from ground-state atoms via cavity quantum electrodynamics. *Phys. Rev. Lett.*, 91:243004, Dec 2003.

[70] Alexey Belyanin, Vitaly V. Kocharovsky, Federico Capasso, Edward Fry, M. Suhail Zubairy, and Marlan O. Scully. Quantum electrodynamics of accelerated atoms in free space and in cavities. *Phys. Rev. A*, 74:023807, Aug 2006.

[71] Marlan O. Scully, Stephen Fulling, David M. Lee, Don N. Page, Wolfgang P. Schleich, and Anatoly A. Svidzinsky. Quantum optics approach to radiation from atoms falling into a black hole. *Proceedings of the National Academy of Sciences*, 115(32):8131–8136, 2018.

[72] Douglas Singleton and Steve Wilburn. Hawking radiation, unruh radiation, and the equivalence principle. *Phys. Rev. Lett.*, 107:081102, Aug 2011.

[73] M. Gell-Mann and F. E. Low. Quantum electrodynamics at small distances. *Physical Review*, 95(5):1300–1312, sep 1954.

[74] Leo P. Kadanoff. Scaling laws for ising models near T_c. *Physics Physique Fizika*, 2:263–272, Jun 1966.

[75] Kenneth G. Wilson. Renormalization group and critical phenomena. i. renormalization group and the kadanoff scaling picture. *Physical Review B*, 4(9):3174–3183, nov 1971.

[76] Kenneth G. Wilson. Renormalization group and critical phenomena. II. phase-space cell analysis of critical behavior. *Physical Review B*, 4(9):3184–3205, nov 1971.

[77] Kenneth G. Wilson and Michael E. Fisher. Critical exponents in 3.99 dimensions. *Physical Review Letters*, 28(4):240–243, jan 1972.

[78] Morgan K. Grover, Leo P. Kadanoff, and Franz J. Wegner. Critical exponents for the heisenberg model. *Physical Review B*, 6(1):311–313, jul 1972.

[79] Kenneth G. Wilson and J. Kogut. The renormalization group and the ϵ expansion. *Physics Reports*, 12(2):75–199, 1974.

[80] Steven Weinberg. Critical phenomena for field theorists. In *Understanding the Fundamental Constituents of Matter*, pages 1–52. Springer US, 1978.

[81] Christof Wetterich. Exact evolution equation for the effective potential. *Physics Letters B*, 301(1):90–94, feb 1993.

[82] TIM R. MORRIS. The exact renormalization group and approximate solutions. *International Journal of Modern Physics A*, 09(14):2411–2449, jun 1994.

[83] M. Reuter and C. Wetterich. Effective average action for gauge theories and exact evolution equations. *Nuclear Physics B*, 417(1-2):181–214, apr 1994.

[84] O. Lauscher and M. Reuter. Ultraviolet fixed point and generalized flow equation of quantum gravity. *Physical Review D*, 65(2), dec 2001.

[85] Djamel Dou and Roberto Percacci. The running gravitational couplings. *Classical and Quantum Gravity*, 15(11):3449–3468, nov 1998.

[86] Roberto Percacci and Daniele Perini. Asymptotic safety of gravity coupled to matter. *Physical Review D*, 68(4), aug 2003.

[87] J.-E. Daum, U. Harst, and M. Reuter. Running gauge coupling in asymptotically safe quantum gravity. *Journal of High Energy Physics*, 2010(1), jan 2010.

[88] U. Harst and M. Reuter. QED coupled to QEG. *Journal of High Energy Physics*, 2011(5), may 2011.

[89] Gaurav Narain and Roberto Percacci. Renormalization group flow in scalar-tensor theories: I. *Classical and Quantum Gravity*, 27(7):075001, mar 2010.

[90] Astrid Eichhorn, Holger Gies, and Michael M. Scherer. Asymptotically free scalar curvature-ghost coupling in quantum einstein gravity. *Phys. Rev. D*, 80:104003, Nov 2009.

[91] Kai Groh and Frank Saueressig. Ghost wavefunction renormalization in asymptotically safe quantum gravity. *Journal of Physics A: Mathematical and Theoretical*, 43(36):365403, jul 2010.

[92] R Percacci. Renormalization group flow of weyl invariant dilaton gravity. *New Journal of Physics*, 13(12):125013, dec 2011.

[93] R. Percaccinand C. Rahmede A. Codello. Ultraviolet properties of f(r)-gravity. *International Journal of Modern Physics A*, 23(01):143–150, jan 2008.

[94] Pedro F. Machado and Frank Saueressig. Renormalization group flow of $f(r)$ gravity. *Phys. Rev. D*, 77:124045, Jun 2008.

[95] Roberto Percacci Alessandro Codello and Christoph Rahmede. Investigating the ultraviolet properties of gravity with a wilsonian renormalization group equation. *Annals of Physics*, 324(2):414–469, feb 2009.

[96] O Lauscher and M Reuter. Is quantum Einstein gravity nonperturbatively renormalizable? *Classical and Quantum Gravity*, 19(3):483–492, jan 2002.

[97] Alessandro Codello and Roberto Percacci. Fixed points of higher-derivative gravity. *Physical Review Letters*, 97(22), nov 2006.

[98] Dario Benedetti, Pedro F. Machado, and Frank Saueressig. Asymptotic safety in higher-derivative gravity. *Modern Physics Letters A*, 24(28):2233–2241, sep 2009.

[99] Martin Reuter and Frank Saueressig. *Quantum Gravity and the Functional Renormalization Group: The Road towards Asymptotic Safety*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2019.

[100] Holger Gies. *Introduction to the Functional RG and Applications to Gauge Theories*, pages 287–348. Springer Berlin Heidelberg, Berlin, Heidelberg, 2012.

[101] Jürgen Berges, Nikolaos Tetradis, and Christof Wetterich. Non-perturbative renormalization flow in quantum field theory and statistical physics. *Physics Reports*, 363(4):223–386, 2002. Renormalization group theory in the new millennium. IV.

[102] Jan M. Pawlowski. Aspects of the functional renormalisation group. *Annals of Physics*, 322(12):2831–2915, 2007.

[103] Janos Polonyi. Lectures on the functional renormalization group method. *Open Physics*, 1(1):1–71, jan 2003.

[104] Oliver J. Rosten. Fundamentals of the exact renormalization group. *Physics Reports*, 511(4):177–272, 2012. Fundamentals of the exact renormalization group.

[105] Martin Reuter and Frank Saueressig. Quantum Einstein gravity. *New Journal of Physics*, 14(5):055022, may 2012.

[106] Daniel F. Litim. Fixed points of quantum gravity. *Phys. Rev. Lett.*, 92:201301, May 2004.

[107] Elisa Manrique and Martin Reuter. Bare action and regularized functional integral of asymptotically safe quantum gravity. *Phys. Rev. D*, 79:025008, Jan 2009.

[108] O. Lauscher and M. Reuter. Towards nonperaturbative renormalizability of quantum Einstein gravity. *International Journal of Modern Physics A*, 17(06n07):993–1002, 2002.

[109] Daniel F. Litim. Optimized renormalization group flows. *Phys. Rev. D*, 64:105007, Oct 2001.

[110] Daniel F. Litim. Optimisation of the exact renormalisation group. *Physics Letters B*, 486(1):92–99, 2000.

[111] Alfio Bonanno and Martin Reuter. Renormalization group improved black hole spacetimes. *Phys. Rev. D*, 62:043008, Jul 2000.

[112] M Reuter and H Weyer. Quantum gravity at astrophysical distances? *Journal of Cosmology and Astroparticle Physics*, 2004(12):001–001, dec 2004.

[113] Aroonkumar Beesham. Bianchi type i cosmological models with variable g and λ. *General Relativity and Gravitation*, 26(2):159–165, feb 1994.

[114] D. Kalligas, P. S. Wesson, and C. W. F. Everitt. Bianchi type i cosmological models with variable g and λ: A comment. *General Relativity and Gravitation*, 27(6):645–650, jun 1995.

[115] J. P. Singh, Anirudh Pradhan, and A. Kumar Singh. Bianchi type-i cosmological models with variable g and λ-term in general relativity. *Astrophysics and Space Science*, 314(1-3):83–88, jan 2008.

[116] Anirudh Pradhan, Rekha Jaiswal, and Rajeev Kumar Khare. Bianchi type-i cosmological models with time dependent q and λ-term in general relativity. *Astrophysics and Space Science*, 343(1):489–497, sep 2012.

[117] D. Kalligas, P. Wesson, and C. W. F. Everitt. Flat FRW models with variable G and λ. *General Relativity and Gravitation*, 24(4):351–357, apr 1992.

[118] Andreas Nink and Martin Reuter. On the physical mechanism underlying asymptotic safety. *Journal of High Energy Physics*, 2013(1), jan 2013.

[119] Daniel F. Litim and Francesco Sannino. Asymptotic safety guaranteed. *Journal of High Energy Physics*, 2014(12), dec 2014.

[120] Eloisa Bentivegna, Alfio Bonanno, and Martin Reuter. Confronting the IR fixed point cosmology with high-redshift observations. *Journal of Cosmology and Astroparticle Physics*, 2004(01):001–001, jan 2004.

[121] Alfio Bonanno, S J Gabriele Gionti, and Alessia Platania. Bouncing and emergent cosmologies from Arnowitt–Deser–Misner RG flows. *Classical and Quantum Gravity*, 35(6):065004, feb 2018.

[122] Alessia Platania and Frank Saueressig. Functional Renormalization Group Flows on Friedman–Lemaître–Robertson–Walker backgrounds. *Found. Phys.*, 48(10):1291–1304, 2018.

[123] Alessia Platania. From renormalization group flows to cosmology. *Frontiers in Physics*, 8, 2020.

[124] Alfio Bonanno and Frank Saueressig. Asymptotically safe cosmology – a status report. *Comptes Rendus Physique*, 18(3):254–264, 2017.

[125] A. Babić, B. Guberina, R. Horvat, and H. Štefančić. Renormalization-group running cosmologies: A scale-setting procedure. *Phys. Rev. D*, 71:124041, Jun 2005.

[126] Silvije Domazet and Hrvoje Štefančić. Renormalization group scale-setting from the action—a road to modified gravity theories. *Classical and Quantum Gravity*, 29(23):235005, oct 2012.

[127] M. Reuter and H. Weyer. Renormalization group improved gravitational actions: A brans-dicke approach. *Phys. Rev. D*, 69:104022, May 2004.

[128] M. Reuter and H. Weyer. Running newton constant, improved gravitational actions, and galaxy rotation curves. *Phys. Rev. D*, 70:124028, Dec 2004.

[129] Pietro Donà, Astrid Eichhorn, and Roberto Percacci. Matter matters in asymptotically safe quantum gravity. *Phys. Rev. D*, 89:084035, Apr 2014.

[130] J. Meibohm, J. M. Pawlowski, and M. Reichert. Asymptotic safety of gravity-matter systems. *Phys. Rev. D*, 93:084035, Apr 2016.

[131] A Bonanno and M Reuter. Entropy signature of the running cosmological constant. *Journal of Cosmology and Astroparticle Physics*, 2007(08):024–024, aug 2007.

[132] A Bonanno and M Reuter. Primordial entropy production and λ-driven inflation from Quantum Einstein Gravity. *Journal of Physics: Conference Series*, 140:012008, nov 2008.

[133] J. A. S. Lima. Thermodynamics of decaying vacuum cosmologies. *Phys. Rev. D*, 54:2571–2577, Aug 1996.

[134] J. A. S. Lima. Cosmologies with photon creation and the 3k relic radiation spectrum. *General Relativity and Gravitation*, 29(7):805–813, jun 1997.

[135] B. Ryden. *Introduction to cosmology.* Cambridge University Press, 1970.

[136] Planck. Planck 2018 results: VI. cosmological parameters. *Astronomy and Astrophysics*, 641, 09 2020.

[137] Harald Fritzsch, Rafael Nunes, and Joan Solà Peracaula. Running vacuum in the universe and the time variation of the fundamental constants of nature. *The European Physical Journal C*, 77, 05 2016.

[138] John D. Barrow, Håvard Bunes Sandvik, and João Magueijo. Behavior of varying-alpha cosmologies. *Phys. Rev. D*, 65:063504, Feb 2002.

[139] J.D. Barrow. Varying alpha. *Annalen der Physik*, 522(3-5):202–210, 2010.

[140] Jean-Philippe Uzan. Varying constants, gravitation and cosmology. *Living Reviews in Relativity*, 14(1), mar 2011.

[141] J. M. Bardeen, B. Carter, and S. W. Hawking. The four laws of black hole mechanics. *Communications in Mathematical Physics*, 31(2):161–170, jun 1973.

[142] Ronald J. Adler, Pisin Chen, and David I. Santiago. The generalized uncertainty principle and black hole remnants. *General Relativity and Gravitation*, 33(12):2101–2108, dec 2001.

[143] Michele Maggiore. A generalized uncertainty principle in quantum gravity. *Physics Letters B*, 304(1):65–69, 1993.

[144] Michele Maggiore. Quantum groups, gravity, and the generalized uncertainty principle. *Phys. Rev. D*, 49:5182–5187, May 1994.

[145] Fabio Scardigli. Generalized uncertainty principle in quantum gravity from micro-black hole gedanken experiment. *Physics Letters B*, 452(1):39–44, 1999.

[146] Sunandan Gangopadhyay, Abhijit Dutta, and Anirban Saha. Generalized uncertainty principle and black hole thermodynamics. *General Relativity and Gravitation*, 46(2), jan 2014.

[147] João Magueijo and Lee Smolin. Lorentz invariance with an invariant energy scale. *Phys. Rev. Lett.*, 88:190403, Apr 2002.

[148] João Magueijo and Lee Smolin. Generalized lorentz invariance with an invariant energy scale. *Phys. Rev. D*, 67:044017, Feb 2003.

[149] Giovanni Amelino-Camelia. Relativity in spacetimes with short-distance structure governed by an observer-independent (planckian) length scale. *International Journal of Modern Physics D*, 11(01):35–59, 2002.

[150] João Magueijo and Lee Smolin. Gravity's rainbow. *Classical and Quantum Gravity*, 21(7):1725–1736, mar 2004.

[151] Maulik K. Parikh and Frank Wilczek. Hawking radiation as tunneling. *Phys. Rev. Lett.*, 85:5042–5045, Dec 2000.

[152] S. Shankaranarayanan. Temperature and entropy of schwarzschild–de sitter spacetime. *Phys. Rev. D*, 67:084026, Apr 2003.

[153] A. Bonanno and M. Reuter. Spacetime structure of an evaporating black hole in quantum gravity. *Phys. Rev. D*, 73:083005, Apr 2006.

[154] S. W. Hawking. Gravitational radiation from colliding black holes. *Phys. Rev. Lett.*, 26:1344–1346, May 1971.

[155] Werner Israel. Event horizons in static vacuum space-times. *Phys. Rev.*, 164:1776–1779, Dec 1967.

[156] C.W. Misner, K.S. Thorne, and J.A. Wheeler. *Gravitation*. Number pt. 3 in Gravitation. W. H. Freeman, 1973.

[157] J. B. Hartle and S. W. Hawking. Path-integral derivation of black-hole radiance. *Phys. Rev. D*, 13:2188–2203, Apr 1976.

[158] Gary William Gibbons, M. J. Perry, and Stephen William Hawking. Black holes and thermal green functions. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 358(1695):467–494, 1978.

[159] G. W. Gibbons and S. W. Hawking. Action integrals and partition functions in quantum gravity. *Phys. Rev. D*, 15:2752–2756, May 1977.

[160] Sunandan Gangopadhyay and Abhijit Dutta. Constraints on rainbow gravity functions from black-hole thermodynamics. *EPL (Europhysics Letters)*, 115(5):50005, sep 2016.

[161] Sunandan Gangopadhyay and Abhijit Dutta. Black hole thermodynamics and generalized uncertainty principle with higher order terms in momentum uncertainty. *Advances in High Energy Physics*, 2018:1–9, jun 2018.

[162] S. W. Hawking and Don N. Page. Thermodynamics of black holes in anti-de sitter space. *Communications in Mathematical Physics*, 87(4):577–588, dec 1983.

[163] J. D. Brown, J. Creighton, and R. B. Mann. Temperature, energy, and heat capacity of asymptotically anti–de sitter black holes. *Phys. Rev. D*, 50:6394–6403, Nov 1994.

[164] James W. York. Black-hole thermodynamics and the euclidean einstein action. *Phys. Rev. D*, 33:2092–2099, Apr 1986.

[165] Richard C. Tolman. On the weight of heat and thermal equilibrium in general relativity. *Phys. Rev.*, 35:904–924, Apr 1930.

[166] G. J. Stephens and B. L. Hu. *International Journal of Theoretical Physics*, 40(12):2183–2200, 2001.

[167] Makoto Natsuume. Ads/cft duality user guide. 903, 09 2014.

[168] Rong-Gen Cai. Gauss-bonnet black holes in ads spaces. *Phys. Rev. D*, 65:084014, Mar 2002.

[169] Yong-Wan Kim, Seung Kook Kim, and Young-Jai Park. Thermodynamic stability of modified schwarzschild–AdS black hole in rainbow gravity. *The European Physical Journal C*, 76(10), oct 2016.

[170] A. Einstein. Die grundlage der allgemeinen relativitätstheorie. *Annalen der Physik*, 354(7):769–822, 1916.

[171] Peter Weiss. Black hole recipe: Slow light, swirl atoms. *Science News*, 157(6):86–86, 2000.

[172] Thomas G. Philbin, Chris Kuklewicz, Scott Robertson, Stephen Hill, Friedrich König, and Ulf Leonhardt. Fiber-optical analog of the event horizon. *Science*, 319(5868):1367–1370, 2008.

[173] Don N. Page. Particle emission rates from a black hole: Massless particles from an uncharged, nonrotating hole. *Phys. Rev. D*, 13:198–206, Jan 1976.

[174] Don N. Page. Particle emission rates from a black hole. II. massless particles from a rotating hole. *Phys. Rev. D*, 14:3260–3273, Dec 1976.

[175] Don N. Page. Particle emission rates from a black hole. iii. charged leptons from a nonrotating hole. *Phys. Rev. D*, 16:2402–2411, Oct 1977.

[176] Stephen A. Fulling. Nonuniqueness of canonical field quantization in riemannian space-time. *Phys. Rev. D*, 7:2850–2862, May 1973.

[177] P C W Davies. Scalar production in schwarzschild and rindler metrics. *Journal of Physics A: Mathematical and General*, 8(4):609–616, apr 1975.

[178] Stephen Hawking and W. Israel. *General Relativity: an Einstein Centenary Survey.* 2010.

[179] Rainer Müller. Decay of accelerated particles. *Phys. Rev. D*, 56:953–960, Jul 1997.

[180] Daniel A. T. Vanzella and George E. A. Matsas. Decay of accelerated protons and the existence of the fulling-davies-unruh effect. *Phys. Rev. Lett.*, 87:151301, Sep 2001.

[181] Luís C. B. Crispino, Atsushi Higuchi, and George E. A. Matsas. The unruh effect and its applications. *Rev. Mod. Phys.*, 80:787–838, Jul 2008.

[182] H. E. Camblong, A. Chakraborty, and C. R. Ordóñez. Near-horizon aspects of acceleration radiation by free fall of an atom into a black hole. *Phys. Rev. D*, 102:085010, Oct 2020.

[183] A. Azizi, H. E. Camblong, A. Chakraborty, C. R. Ordóñez, and M. O. Scully. Acceleration radiation of an atom freely falling into a kerr black hole and near-horizon conformal quantum mechanics. *Phys. Rev. D*, 104:065006, Sep 2021.

[184] A. Azizi, H. E. Camblong, A. Chakraborty, C. R. Ordóñez, and M. O. Scully. Quantum optics meets black hole thermodynamics via conformal quantum mechanics: Ii. thermodynamics of acceleration radiation. *Phys. Rev. D*, 104:084085, Oct 2021.

[185] A. Azizi, H. E. Camblong, A. Chakraborty, C. R. Ordóñez, and M. O. Scully. Quantum optics meets black hole thermodynamics via conformal quantum mechanics. i. master equation for acceleration radiation. *Phys. Rev. D*, 104:084086, Oct 2021.

[186] Carlo Rovelli. Loop quantum gravity. *Living Reviews in Relativity*, 1(1), jan 1998.

[187] S Carlip. Quantum gravity: a progress report. *Reports on Progress in Physics*, 64(8):885–942, jul 2001.

[188] D. Amati, M. Ciafaloni, and G. Veneziano. Can spacetime be probed below the string size? *Physics Letters B*, 216(1):41–47, 1989.

[189] Kenichi Konishi, Giampiero Paffuti, and Paolo Provero. Minimum physical length and the generalized uncertainty principle in string theory. *Physics Letters B*, 234(3):276–284, 1990.

[190] F Rohrlich. The principle of equivalence. *Annals of Physics*, 22(2):169–191, 1963.

[191] M. Pauri and M. Vallisneri. *Foundations of Physics*, 29(10):1499–1520, 1999.

[192] Douglas Singleton and Steve Wilburn. Global versus local — mach's principle versus the equivalence principle. *International Journal of Modern Physics D*, 25(12):1644009, 2016.

[193] Anatoly A. Svidzinsky, Jonathan S. Ben-Benjamin, Stephen A. Fulling, and Don N. Page. Excitation of an atom by a uniformly accelerated mirror through virtual transitions. *Phys. Rev. Lett.*, 121:071301, Aug 2018.

[194] S A Fulling and J H Wilson. The equivalence principle at work in radiation from unaccelerated atoms and mirrors. *Physica Scripta*, 94(1):014004, nov 2018.

[195] Riddhi Chatterjee, Sunandan Gangopadhyay, and A. S. Majumdar. Violation of equivalence in an accelerating atom-mirror system in the generalized uncertainty principle framework. *Phys. Rev. D*, 104:124001, Dec 2021.

[196] Romesh K. Kaul and Parthasarathi Majumdar. Logarithmic correction to the bekenstein-hawking entropy. *Phys. Rev. Lett.*, 84:5255–5257, Jun 2000.

[197] David G. Boulware. Quantum field theory in schwarzschild and rindler spaces. *Phys. Rev. D*, 11:1404–1423, Mar 1975.